Call-by-Name and Call-by-Value in Normal Modal Logic

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Abstract. This paper provides a call-by-name and a call-by-value calculus, both of which have a Curry-Howard correspondence to the minimal normal logic **K**. The calculi are extensions of the $\lambda\mu$-calculi, and their semantics are given by CPS transformations into a calculus corresponding to the intuitionistic fragment of **K**. The duality between call-by-name and call-by-value with modalities is investigated in our calculi.

1 Introduction

Modal logics have a long history since logics with strict implications, and are now widely accepted both theoretically and practically. Especially, studies of modal logics by Kripke semantics [18] are quite active and a large number of results exist, for example, [7] is a textbook about such studies. Since Kripke semantics concern only provability, equality on proofs is less studied on modal logics compared with traditional logics.

It is well-known that the intuitionistic propositional logic exactly corresponds to the simply typed λ-calculus: formulae as types and proofs as terms. Such a correspondence is called a Curry-Howard correspondence after Howard's work [15]. A Curry-Howard correspondence enables us to study equality on proofs computationally. Though the correspondence can be extended to higher-order and predicate logics as shown in [3], we investigate only propositional logics in this paper. The aim of this study is to give a proper calculus that have a Curry-Howard correspondence to the modal logic **K**. Through a Curry-Howard correspondence, any type system can be regarded as a logic by forgetting terms. In this sense, modal logics are contributing to practical studies for programming languages, *e.g.*, staged computations [8] and information flow analysis [23]. Since **K** is known as a minimal modal logic, this paper focuses on **K** rather than **S4**.

Before defining a calculus for **K**, we consider the intuitionistic fragment of **K**, which is called **IK** in this paper. In Section 2, the calculus for **IK** is defined as a refinement of Bellin *et al.*'s calculus [4] rather than Martini and Masini's[22]. Our calculus is sound and complete for the categorical semantics given in [4]. The study [19] about simply typed λ-calculus and cartesian closed categories is a typical study of categorical semantics. Categorical semantics of modal logics are studied by Bierman and de Paiva, and Bellin *et al.* in [6] and [4]. Their semantics are based on studies about semantics of linear logics (*e.g.*, [29] and [5]) since the exponential of the linear logic [12] is a kind of **S4** modality.

A Curry-Howard correspondence between the classical propositional logic and the λ-calculus with continuations was provided in [13] by Griffin. Parigot has proposed the $\lambda\mu$-calculus as a calculus for the classical logic in [26]. Now, kinds of $\lambda\mu$-calculi exist and some of them are defined by CPS transformations. A CPS transformation was originally introduced in [11], and the relation between call-by-value and CPS semantics was first studied by Plotkin in [27]. De Groote defines a CPS transformation on a call-by-name $\lambda\mu$-calculus in [9], but in this paper, we adopt Selinger's CPS transformation [30], which is an extension of Hofmann and Streicher's [14]. In Section 3, we provide a call-by-name $\lambda\mu$-calculus with a box modality, which has a Curry-Howard correspondence to **K**, by the CPS semantics into the calculus for **IK** defined in Section 2. A call-by-value $\lambda\mu$-calculus is provided in [25] by Ong and Stewart. We define a call-by-value calculus for **K** also as an extension of Selinger's call-by-value $\lambda\mu$-calculus [30] via the CPS transformation in Section 4.

The duality between call-by-name and call-by-value is an important property of the classical logic. The duality on a programming language with first-class continuations was first formalized by Filinski in [10]. It has been formalized on the $\lambda\mu$-calculi in [30] by Selinger, and reformulated as sequent calculi in [34] by Wadler. In [16], the duality is developed with recursion by the author. In Section 5, we study such duality on the classical modal logic **K**.

In addition, we investigate the logic **S4** with the CPS semantics. It is shown in Section 6 that a diamond modality is a monad in call-by-name **S4**.

Notations

We introduce notations specific to this paper.

- The symbol "$\equiv$" denotes the α-equivalence.
- We may omit superscripted and subscripted types if they are trivial.
- A notation "$\overrightarrow{M}$" is used for a sequence of meta-variables "$M_1, \ldots, M_n$". Hence, an expression "$\overrightarrow{M}, \overrightarrow{N}$" stands for the concatenation of $\overrightarrow{M}$ and $\overrightarrow{N}$.
- For a unary operator $\Phi(-)$, we write "$\Phi(\overrightarrow{M})$" for "$\Phi(M_1), \ldots, \Phi(M_n)$".
- We write "$\overrightarrow{N}(\theta \overrightarrow{x}.\, M)$" for "$N_1(\theta x_1. \cdots N_n(\theta x_n.\, M) \cdots)$" and "$[\overrightarrow{a}](\theta \overrightarrow{x}.\, M)$" for "$[a_1](\theta x_1. \cdots [a_n](\theta x_n.\, M) \cdots)$", where θ is λ or μ.
- We write "$\neg\tau$" for "$\tau \to \bot$".

2 Calculus for Intuitionistic Normal Modal Logic

In this section, we study the intuitionistic modal logic **IK**. Intuitionism of a diamond modality is not trivial, for example, [33] gives an account of it, but this section focuses on the box fragment of **IK**. We call also this fragment itself **IK** in this paper. A diamond modality is investigated in a classical logic after the next section.

It is well-known that the λ-calculus with conjunctions and disjunctions exactly corresponds to the intuitionistic propositional logic. Therefore, we extend

Table 1. Typing rules of $\lambda\Box$-calculus

$$\frac{}{\Gamma \vdash \alpha^\tau : \tau} \qquad \frac{}{\Gamma, x:\tau, \Gamma' \vdash x : \tau}$$

$$\frac{}{\Gamma \vdash \langle\,\rangle : \top} \qquad \frac{}{\Gamma \vdash [\,]_\tau : \bot \to \tau}$$

$$\frac{\Gamma, x:\sigma \vdash M : \tau}{\Gamma \vdash \lambda x^\sigma.\, M : \sigma \to \tau} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \quad \Gamma \vdash M_2 : \tau_2}{\Gamma \vdash \langle M_1, M_2\rangle : \tau_1 \wedge \tau_2} \qquad \frac{\Gamma \vdash M : \tau_1 \wedge \tau_2}{\Gamma \vdash \pi_j M : \tau_j}$$

$$\frac{\Gamma, x_1:\sigma_1 \vdash M_1 : \tau \quad \Gamma, x_2:\sigma_2 \vdash M_2 : \tau}{\Gamma \vdash [\lambda x_1^{\sigma_1}.\, M_1, \lambda x_2^{\sigma_2}.\, M_2] : \sigma_1 \vee \sigma_2 \to \tau} \qquad \frac{\Gamma \vdash M : \tau_j}{\Gamma \vdash \iota_j M : \tau_1 \vee \tau_2}$$

$$\frac{x_1:\sigma_1, \ldots, x_n:\sigma_n \vdash M : \tau \quad \Gamma \vdash N_1 : \Box\sigma_1 \quad \cdots \quad \Gamma \vdash N_n : \Box\sigma_n}{\Gamma \vdash \texttt{box}\,\langle x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}\rangle\ \texttt{be}\ \langle N_1, \ldots, N_n\rangle\ \texttt{in}\ M : \Box\tau}$$

the λ-calculus with a box construct. Our calculus, called the $\lambda\Box$-calculus, is a refinement of Bellin *et al.*'s calculus given in [4]. The difference from Bellin *et al.*'s is discussed in [17].

Definition 1. *The $\lambda\Box$-calculus is defined as follows. Types τ and terms M are defined by*

$$\begin{aligned}
\tau &::= p \mid \tau \to \tau \mid \top \mid \tau \wedge \tau \mid \bot \mid \tau \vee \tau \mid \Box\tau \\
M &::= \alpha^\tau \mid x \mid \lambda x^\tau.\, M \mid MM \mid \langle\,\rangle \mid \langle M, M\rangle \mid \pi_1 M \mid \pi_2 M \\
&\quad\mid [\,]_\tau \mid [\lambda x^\tau.\, M, \lambda x^\tau.\, M] \mid \iota_1 M \mid \iota_2 M \\
&\quad\mid \texttt{box}\,\langle x^\tau, \ldots, x^\tau\rangle\ \texttt{be}\ \langle M, \ldots, M\rangle\ \texttt{in}\ M
\end{aligned}$$

where p, c, and x range over type constants, constants, and variables, respectively. A variable x_i occurring freely in M is bound in the term $\texttt{box}\,\langle x_1, \ldots, x_n\rangle\texttt{be}$ $\langle N_1, \ldots, N_n\rangle$ $\texttt{in}$ M*. The typing rules are given in Table 1. The equality is defined by the axioms given in Table 2. In the table, an equation $M = N$ means that two derivable judgments $\Gamma \vdash M : \tau$ and $\Gamma \vdash N : \tau$ are equal for any Γ and τ. A theory including the equality of the $\lambda\Box$-calculus is called a $\lambda\Box$-theory.*

Note that all free variables of M are covered by $\overrightarrow{x}$ when the term $\texttt{box}\,\langle\overrightarrow{x}\rangle\,\texttt{be}$ $\langle\overrightarrow{N}\rangle$ $\texttt{in}$ M is typable. It means that each box encloses a proof.

Since the $\lambda\Box$-calculus has essentially the same syntax as the calculus defined in [4], we can show that our calculus corresponds to the intuitionistic modal logic.

Let **IK** be an intuitionistic Hilbert system with the axiom $\Box(\sigma \to \tau) \to \Box\sigma \to \Box\tau$ and the box inference rule. The $\lambda\Box$-calculus can be regarded as a natural deduction by forgetting terms. It is shown as follows that our logic is equivalent to **IK** with respect to provability. The box inference rule of **IK** is simulated by

$$\frac{\vdash M : \tau}{\vdash \texttt{box}\,\langle\,\rangle\ \texttt{be}\ \langle\,\rangle\ \texttt{in}\ M : \Box\tau}$$

Table 2. Axioms of $\lambda\Box$-calculus

$$
\begin{array}{ll}
(\lambda x.\, M)N = M\{N/x\} & \\
\lambda x.\, Mx = M & \text{if } x \notin \mathtt{FV}(M) \\
\langle\,\rangle = M & \\
\pi_j \langle M_1, M_2 \rangle = M_j & \\
\langle \pi_1 M, \pi_2 M \rangle = M & \\
[\,] = M & \\
[\lambda x_1.\, M_1, \lambda x_2.\, M_2](\iota_j N) = (\lambda x_j.\, M_j)N & \\
[\lambda x_1.\, M(\iota_1 x_1), \lambda x_2.\, M(\iota_2 x_2)] = M & \text{if } x_1, x_2 \notin \mathtt{FV}(M) \\
\mathtt{box}\,\langle x \rangle\ \mathtt{be}\ \langle M \rangle\ \mathtt{in}\ x = M & \\
\mathtt{box}\,\langle \overrightarrow{w}, x, \overrightarrow{z} \rangle\ \mathtt{be}\ \langle \overrightarrow{Q}, \mathtt{box}\,\langle \overrightarrow{y} \rangle\ \mathtt{be}\ \langle \overrightarrow{L} \rangle\ \mathtt{in}\ N, \overrightarrow{P} \rangle\ \mathtt{in}\ M & \\
\quad = \mathtt{box}\,\langle \overrightarrow{w}, \overrightarrow{y}, \overrightarrow{z} \rangle\ \mathtt{be}\ \langle \overrightarrow{Q}, \overrightarrow{L}, \overrightarrow{P} \rangle\ \mathtt{in}\ M\{N/x\} & \text{if } |\overrightarrow{w}| = |\overrightarrow{Q}|
\end{array}
$$

and the distributivity is realized by the judgment

$$\vdash\ \lambda f'.\, \lambda x'.\, \mathtt{box}\,\langle f, x \rangle\ \mathtt{be}\ \langle f', x' \rangle\ \mathtt{in}\ fx : \Box(\sigma \to \tau) \to \Box\sigma \to \Box\tau$$

in our logic. Conversely, **IK** simulates our typing rule as

$$
\dfrac{\dfrac{\dfrac{\vdots}{\dfrac{\varGamma \vdash \Box(\sigma_n \to \tau)}{\varGamma \vdash \Box\sigma_n \to \Box\tau}}\qquad \varGamma \vdash \Box\sigma_n}{}}{\varGamma \vdash \Box\tau}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\sigma_1, \ldots, \sigma_n \vdash \tau}{\vdash \sigma_1 \to \cdots \to \sigma_n \to \tau}}{\vdash \Box(\sigma_1 \to \cdots \to \sigma_n \to \tau)}}{\varGamma \vdash \Box(\sigma_1 \to \cdots \to \sigma_n \to \tau)}}{\varGamma \vdash \Box\sigma_1 \to \Box(\sigma_2 \to \cdots \to \sigma_n \to \tau)} \qquad \varGamma \vdash \Box\sigma_1}{\varGamma \vdash \Box(\sigma_2 \to \cdots \to \sigma_n \to \tau)}
\quad \cdots \quad
\dfrac{\dfrac{\dfrac{\vdots}{\varGamma \vdash \Box(\sigma_n \to \tau)}}{\varGamma \vdash \Box\sigma_n \to \Box\tau} \qquad \varGamma \vdash \Box\sigma_n}{\varGamma \vdash \Box\tau}\ .
$$

We can also show more directly that our logic corresponds to the sequent calculus formulation of **IK** proposed in [35].

According to the above encoding, it is not trivial whether an exchange rule commutes with a box operation. Therefore, we distinguish such symmetricity from other axioms although it is common to consider proofs up to exchanges.

Definition 2. *In a $\lambda\Box$-theory, $\Box$ is symmetric if the equation*

$$
\begin{array}{ll}
\mathtt{box}\,\langle \overrightarrow{w}, x, y, \overrightarrow{z} \rangle\ \mathtt{be}\ \langle \overrightarrow{Q}, N, L, \overrightarrow{P} \rangle\ \mathtt{in}\ M & \\
\quad = \mathtt{box}\,\langle \overrightarrow{w}, y, x, \overrightarrow{z} \rangle\ \mathtt{be}\ \langle \overrightarrow{Q}, L, N, \overrightarrow{P} \rangle\ \mathtt{in}\ M & \textit{if } |\overrightarrow{w}| = |\overrightarrow{Q}|
\end{array}
$$

is satisfied.

Our axiomatization is justified logically via the Curry-Howard correspondence: the equation

$$\texttt{box}\,\langle x\rangle\,\texttt{be}\,\langle M\rangle\,\texttt{in}\,x = M$$

says that a trivial boxed proof can be removed, and the equation

$$\begin{aligned}&\texttt{box}\,\langle \overrightarrow{w}, x, \overrightarrow{z}\rangle\,\texttt{be}\,\langle \overrightarrow{Q}, \texttt{box}\,\langle \overrightarrow{y}\rangle\,\texttt{be}\,\langle \overrightarrow{L}\rangle\,\texttt{in}\,N, \overrightarrow{P}\rangle\,\texttt{in}\,M\\ &= \texttt{box}\,\langle \overrightarrow{w}, \overrightarrow{y}, \overrightarrow{z}\rangle\,\texttt{be}\,\langle \overrightarrow{Q}, \overrightarrow{L}, \overrightarrow{P}\rangle\,\texttt{in}\,M\{N/x\}\end{aligned}$$

says that adjacent boxes can be combined into one box. In addition, Abe characterizes the $\lambda\Box$-calculus by a standard translation into the intuitionistic predicate logic in [1]. Computational meaning of the $\lambda\Box$-calculus is shown as follows.

We consider categorical models of **IK** along the line of [4]. Because Kripke semantics cover provability but not proofs themselves, they are not suitable for our aim. Since our calculus is an extension of the simply typed λ-calculus, a model of the $\lambda\Box$-calculus should be a cartesian closed category with finite coproducts. ([19] provides a deep analysis of the λ-calculus and cartesian closed categories.) In addition, the $\lambda\Box$-calculus requires a modality. Roughly speaking, the modality behaves like a functor and is characterized by the axiom $\Box\sigma \wedge \Box\tau \to \Box(\sigma\wedge\tau)$, which is an adjoint of $\Box(\sigma\to\tau)\to\Box\sigma\to\Box\tau$. Assuming that this axiom is parametric, the modality is just *a monoidal endofunctor* with respect to cartesian products. (Fundamental properties of monoidal categories are found in [20].) Hence, a model of **IK** is naturally considered a cartesian closed category with a lax monoidal endofunctor with respect to cartesian products.

An interpretation is given in the usual manner: a type is interpreted as an object and a judgment is interpreted as a morphism. Let a bicartesian closed category $\mathcal{C}$ have a monoidal endofunctor $\Box$ with a natural transformation $\texttt{m}_{A,B} : \Box A \times \Box B \to \Box(A\times B)$ and $\texttt{m}_\top : \top \to \Box\top$. We write $\texttt{m}^*$ as a composite of $\texttt{m}_{A,B}$'s or $\texttt{m}_\top$. An interpretation $[\![-]\!]$ of the $\lambda\Box$-calculus into $\mathcal{C}$ is defined inductively by $[\![\Box\sigma]\!] = \Box[\![\sigma]\!]$ and

$$\begin{aligned}&[\![\Gamma \vdash \texttt{box}\,\langle \overrightarrow{x}\rangle\,\texttt{be}\,\langle \overrightarrow{N}\rangle\,\texttt{in}\,M : \Box\tau]\!]\\ &\quad = [\![\Gamma]\!] \xrightarrow{\langle g_1,\ldots,g_n\rangle} \Box[\![\sigma_1]\!]\times\cdots\times\Box[\![\sigma_n]\!] \xrightarrow{\texttt{m}^*} \Box([\![\sigma_1]\!]\times\cdots\times[\![\sigma_n]\!]) \xrightarrow{\Box f} \Box[\![\tau]\!]\\ &\quad\text{where } g_i = [\![\Gamma \vdash N_i : \Box\sigma_i]\!] \text{ and } f = [\![x_1:\sigma_1,\ldots,x_n:\sigma_n \vdash M:\tau]\!].\end{aligned}$$

Theorem 3. *The $\lambda\Box$-calculus is sound and complete for the class of bicartesian closed categories with monoidal endofunctors.*
The $\lambda\Box$-calculus with symmetric $\Box$ is sound and complete for the class of bicartesian closed categories with symmetric monoidal endofunctors.

Proof. The soundness is shown by induction as usual. The former box axiom holds because a functor preserves identities. The latter box axiom holds because of the naturality of $\texttt{m}$.

The completeness is shown by construction of the term model. The functor conditions are derived from the former box axiom and a special case of the latter axiom. A natural transformation is given by $\lambda x.(\texttt{box}\,\langle y_1, y_2\rangle\ \texttt{be}\ \langle \pi_1 x, \pi_2 x\rangle\ \texttt{in}\ \langle y_1, y_2\rangle)$ and its properties follow from the latter box axiom. □

In fact, the term model in the above proof is an initial model, so we can get a result about internal languages in addition. Because of the space limitation, we omit a discussion about the category of λ□-theories along the line of [19]. The notion of equivalence on categories with monoidal endofunctors follows [21].

Definition 4. *The internal language of a bicartesian closed category $\mathcal{C}$ with a monoidal endofunctor is a λ□-theory whose type constants consist of objects of $\mathcal{C}$ and whose constants consist of morphisms of $\mathcal{C}$ such that the canonical interpretation is sound and complete.*

Proposition 5. *A bicartesian closed category with a monoidal endofunctor is equivalent to the term model of its internal language.*

One might expect a monoidal endofunctor to be strong, *i.e.*, to preserve products, but it can be reminded of the intuitionistic modal logic **IS4**. A model of the box fragment of **IS4** is a cartesian closed category with a monoidal comonad as mentioned by Bierman and de Paiva in [6]. Here, a monoidal comonad is a lax monoidal functor but not a strong monoidal functor in general. If the modality of **IK** is required to be strong monoidal, a model of **IS4** cannot be a model of **IK**. Therefore, we do not require the modality to be strong monoidal. Nevertheless it is possible to consider a strong monoidal functor in our calculus.

Definition 6. *In a λ□-theory, □ is strong if the equations*

$$\begin{aligned}
&\texttt{box}\,\langle \overrightarrow{w}, x, \overrightarrow{z}\rangle\ \texttt{be}\ \langle \overrightarrow{Q}, N, \overrightarrow{P}\rangle\ \texttt{in}\ M \\
&\quad = \texttt{box}\,\langle \overrightarrow{w}, \overrightarrow{z}\rangle\ \texttt{be}\ \langle \overrightarrow{Q}, \overrightarrow{P}\rangle\ \texttt{in}\ M && \text{if } |\overrightarrow{w}| = |\overrightarrow{Q}| \\
&\texttt{box}\,\langle \overrightarrow{w}, x, y, \overrightarrow{z}\rangle\ \texttt{be}\ \langle \overrightarrow{Q}, N, N, \overrightarrow{P}\rangle\ \texttt{in}\ M \\
&\quad = \texttt{box}\,\langle \overrightarrow{w}, x, \overrightarrow{z}\rangle\ \texttt{be}\ \langle \overrightarrow{Q}, N, \overrightarrow{P}\rangle\ \texttt{in}\ M\{x/y\} && \text{if } |\overrightarrow{w}| = |\overrightarrow{Q}|
\end{aligned}$$

are satisfied.

The soundness and completeness of the λ□-calculus with strong □ are proved in the same way as Theorem 3.

Theorem 7. *The λ□-calculus with strong (resp. strong symmetric) □ is sound and complete for the class of bicartesian closed categories with strong (resp. strong symmetric) monoidal functors.*

It is also possible to define the linear version of the λ□-calculus if we restrict occurrence of every free variable to only once. Because in fact the proof of Theorem 3 does not depend on properties of cartesian products, also the linear calculus enjoys the theorem: the linear λ□-calculus is sound and complete for the class of monoidal closed categories with monoidal endofunctors.

Remark 8. This paper is overall motivated by equality on proofs and does not address reductions, but the author proposes a reduction system for the implication fragment of the λ□-calculus in [17]. The strong normalizability, the confluency, and the subformula property of the calculus has been proved in [17].

Table 3. Typing rules of $\lambda\mu\Box$-calculus

$$\frac{}{\Gamma \vdash \alpha^\tau : \tau \mid \Delta} \qquad \frac{}{\Gamma, x:\tau, \Gamma' \vdash x : \tau \mid \Delta} \qquad \frac{}{\Gamma \vdash \langle\,\rangle : \top \mid \Delta}$$

$$\frac{\Gamma, x:\sigma \vdash M : \tau \mid \Delta}{\Gamma \vdash \lambda x^\sigma.\, M : \sigma \to \tau \mid \Delta} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \mid \Delta \quad \Gamma \vdash N : \sigma \mid \Delta}{\Gamma \vdash MN : \tau \mid \Delta}$$

$$\frac{\Gamma \vdash M_1 : \tau_1 \mid \Delta \quad \Gamma \vdash M_2 : \tau_2 \mid \Delta}{\Gamma \vdash \langle M_1, M_2\rangle : \tau_1 \wedge \tau_2 \mid \Delta} \qquad \frac{\Gamma \vdash M : \tau_1 \wedge \tau_2 \mid \Delta}{\Gamma \vdash \pi_j M : \tau_j \mid \Delta}$$

$$\frac{\Gamma \vdash M : \bot \mid a:\tau, \Delta}{\Gamma \vdash \mu a^\tau.\, M : \tau \mid \Delta} \qquad \frac{\Gamma \vdash M : \tau \mid \Delta, a:\tau, \Delta'}{\Gamma \vdash [a]M : \bot \mid \Delta, a:\tau, \Delta'}$$

$$\frac{\Gamma \vdash M : \bot \mid a_1:\tau_1, a_2:\tau_2, \Delta}{\Gamma \vdash \mu(a_1^{\tau_1}, a_2^{\tau_2}).\, M : \tau_1 \vee \tau_2 \mid \Delta} \qquad \frac{\Gamma \vdash M : \tau_1 \vee \tau_2 \mid \Delta, a_1:\tau_1, a_2:\tau_2, \Delta'}{\Gamma \vdash [a_1, a_2]M : \bot \mid \Delta, a_1:\tau_1, a_2:\tau_2, \Delta'}$$

$$\frac{x_1:\sigma_1, \ldots, x_n:\sigma_n \vdash M : \tau \mid \quad \Gamma \vdash N_1 : \Box\sigma_1 \mid \Delta \quad \cdots \quad \Gamma \vdash N_n : \Box\sigma_n \mid \Delta}{\Gamma \vdash \texttt{box}\,\langle x_1^{\sigma_1}, \ldots, x_n^{\sigma_n}\rangle\ \texttt{be}\ \langle N_1, \ldots, N_n\rangle\ \texttt{in}\ M : \Box\tau \mid \Delta}$$

3 Call-by-Name Calculus for Normal Modal Logic

We have defined a calculus for the intuitionistic modal logic in the previous section. This section provides a calculus corresponding to the classical normal modal logic **K**. Our calculus is defined as an extension of Selinger's version [30] of the $\lambda\mu$-calculus, which has a Curry-Howard correspondence to the classical logic. The semantics of the calculus is given by a CPS transformation to the $\lambda\Box$-calculus.

For abbreviation, we may write $\lambda\langle x_1, x_2\rangle.\, M$ for $\lambda y.\, (\lambda x_1.\, \lambda x_2.\, M)(\pi_1 y)(\pi_2 y)$ and $[M_1, M_2]$ for $[\lambda y_1.\, M_1 y_1, \lambda y_2.\, M_2 y_2]$ in the $\lambda\Box$-calculus.

Definition 9. *The call-by-name $\lambda\mu\Box$-calculus is defined as follows. Types τ and terms M are defined by*

$$\begin{aligned}
\tau &::= p \mid \tau \to \tau \mid \top \mid \tau \wedge \tau \mid \bot \mid \tau \vee \tau \mid \Box\tau \\
M &::= \alpha^\tau \mid x \mid \lambda x^\tau.\, M \mid MM \mid \langle\,\rangle \mid \langle M, M\rangle \mid \pi_1 M \mid \pi_2 M \\
&\quad\ \mid \mu a^\tau.\, M \mid [a]M \mid \mu(a^\tau, a^\tau).\, M \mid [a, a]M \\
&\quad\ \mid \texttt{box}\,\langle x^\tau, \ldots, x^\tau\rangle\ \texttt{be}\ \langle M, \ldots, M\rangle\ \texttt{in}\ M
\end{aligned}$$

where p, c, x, and a range over type constants, constants, variables, and control variables, respectively. The typing rules are given in Table 3. The equality is defined by the transformation $[\![-]\!]_{\mathrm{n}}$ to the $\lambda\Box$-calculus with a type constant $\texttt{R}$ *given in Table 4. We write $M =_{\mathrm{n}} N$ for $[\![M]\!]_{\mathrm{n}} = [\![N]\!]_{\mathrm{n}}$ when M and N have the same type.*

A typing derivation of $x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau \mid a_1 : \tau_1, \ldots, a_m : \tau_m$ in the $\lambda\mu\Box$-calculus is regarded as a natural deduction style derivation of $\sigma_1, \ldots, \sigma_n, \neg\tau_1, \ldots, \neg\tau_m \vdash \tau$. By the same reason as in the intuitionistic case, it can be seen that the $\lambda\mu\Box$-calculus corresponds to the classical modal logic **K** with respect to provability.

Table 4. CBN CPS transformation

$$p^\circ \equiv p \qquad (\Box\tau)^\circ \equiv \Box(\tau^\circ \to \mathtt{R}) \to \mathtt{R}$$
$$(\neg\sigma)^\circ \equiv \sigma^\circ \to \mathtt{R} \qquad (\sigma \to \tau)^\circ \equiv (\sigma^\circ \to \mathtt{R}) \wedge \tau^\circ \text{ if } \tau \not\equiv \bot$$
$$\top^\circ \equiv \bot \qquad (\tau_1 \wedge \tau_2)^\circ \equiv {\tau_1}^\circ \vee {\tau_2}^\circ$$
$$\bot^\circ \equiv \top \qquad (\tau_1 \vee \tau_2)^\circ \equiv {\tau_1}^\circ \wedge {\tau_2}^\circ$$

$$\begin{aligned}
[\![\alpha]\!]_{\mathrm{n}} &\equiv \alpha \\
[\![x]\!]_{\mathrm{n}} &\equiv x \\
[\![\lambda x.\, M]\!]_{\mathrm{n}} &\equiv \lambda x.\, [\![M]\!]_{\mathrm{n}}\langle\,\rangle \;\text{ if }\; M : \bot \\
&\equiv \lambda\langle x, k\rangle.\, [\![M]\!]_{\mathrm{n}} k \;\text{ o.w.} \\
[\![MN]\!]_{\mathrm{n}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{n}}[\![N]\!]_{\mathrm{n}} \;\text{ if }\; MN : \bot \\
&\equiv \lambda k.\, [\![M]\!]_{\mathrm{n}}\langle [\![N]\!]_{\mathrm{n}}, k\rangle \;\text{ o.w.} \\
[\![\langle\,\rangle]\!]_{\mathrm{n}} &\equiv [\,] \\
[\![\langle M_1, M_2\rangle]\!]_{\mathrm{n}} &\equiv [[\![M_1]\!]_{\mathrm{n}}, [\![M_2]\!]_{\mathrm{n}}] \\
[\![\pi_j M]\!]_{\mathrm{n}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{n}}(\iota_j k) \\
[\![\mu a.\, M]\!]_{\mathrm{n}} &\equiv \lambda a.\, [\![M]\!]_{\mathrm{n}}\langle\,\rangle \\
[\![[a]M]\!]_{\mathrm{n}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{n}} a \\
[\![\mu(a_1, a_2).\, M]\!]_{\mathrm{n}} &\equiv \lambda\langle a_1, a_2\rangle.\, [\![M]\!]_{\mathrm{n}}\langle\,\rangle \\
[\![[a_1, a_2]M]\!]_{\mathrm{n}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{n}}\langle a_1, a_2\rangle \\
[\![\mathtt{box}\,\langle\overrightarrow{x}\rangle\,\mathtt{be}\,\langle\overrightarrow{N}\rangle\,\mathtt{in}\,M]\!]_{\mathrm{n}} &\equiv \lambda k.\, [\![\overrightarrow{N}]\!]_{\mathrm{n}}(\lambda\overrightarrow{x'}.\, k(\mathtt{box}\,\langle\overrightarrow{x}\rangle\,\mathtt{be}\,\langle\overrightarrow{x'}\rangle\,\mathtt{in}\,[\![M]\!]_{\mathrm{n}}))
\end{aligned}$$

$$\frac{x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash M : \tau \mid a_1 : \tau_1, \ldots, a_m : \tau_m}{x_1 : {\sigma_1}^\circ \to \mathtt{R}, \ldots, x_n : {\sigma_n}^\circ \to \mathtt{R}, a_1 : {\tau_1}^\circ, \ldots, a_m : {\tau_m}^\circ \vdash [\![M]\!]_{\mathrm{n}} : \tau^\circ \to \mathtt{R}}$$

Unlike the intuitionistic case, call-by-name classical disjunctions are not coproducts. (Our formulation of disjunctions is based on Selinger's [30], but it is possible to define the calculus along the line of [28].) Instead of case functions and injections, we use the syntax sugar

$$\begin{aligned}
&[\lambda x_1^{\sigma_1}.\, M_1, \lambda x_2^{\sigma_2}.\, M_2] \\
&\quad \equiv \lambda x^{\sigma_1 \vee \sigma_2}.\, \mu b^{\tau}.\, [b]((\lambda x_1^{\sigma_1}.\, M_1)(\mu a_1^{\sigma_1}.\, [b]((\lambda x_2^{\sigma_2}.\, M_2)(\mu a_2^{\sigma_2}.\, [a_1, a_2]x)))) \\
&\iota_j M \equiv \mu(a_1^{\tau_1}, a_2^{\tau_2}).\, [a_j]M \;\text{ where }\; a_1, a_2 \notin \mathtt{FV}(M).
\end{aligned}$$

These abbreviations are applied to also the call-by-value $\lambda\mu\Box$-calculus given in the next section.

Remark 10. In the definition of the CPS transformation, the cases of abstractions and applications depend on the types, but such dependency is not essential for the semantics. It is just a technical requirement for the syntactic duality shown in Section 5.

The equality is defined by the CPS transformation, so it is not trivial which kind of equation holds. We show some equations which hold in the $\lambda\mu\Box$-calculus but do not hold in the ordinary $\lambda\mu$-calculus. (Of course, equations that hold in the ordinary $\lambda\mu$-calculus hold in the $\lambda\mu\Box$-calculus.)

Proposition 11. *The following equations hold in the call-by-name $\lambda\mu\Box$-calculus.*

$$\begin{aligned}
&\texttt{box}\,\langle x\rangle\;\texttt{be}\;\langle M\rangle\;\texttt{in}\;x =_{\mathrm{n}} M\\
&\texttt{box}\,\langle \overrightarrow{w}, x, \overrightarrow{z}\rangle\;\texttt{be}\;\langle \overrightarrow{Q}, \texttt{box}\,\langle\overrightarrow{y}\rangle\;\texttt{be}\;\langle\overrightarrow{L}\rangle\;\texttt{in}\;N, \overrightarrow{P}\rangle\;\texttt{in}\;M\\
&\quad =_{\mathrm{n}} \texttt{box}\,\langle \overrightarrow{w}, \overrightarrow{y}, \overrightarrow{z}\rangle\;\texttt{be}\;\langle \overrightarrow{Q}, \overrightarrow{L}, \overrightarrow{P}\rangle\;\texttt{in}\;M\{N/x\} \qquad \textit{if } |\overrightarrow{w}| = |\overrightarrow{Q}|\\
&\texttt{box}\,\langle \overrightarrow{x}\rangle\;\texttt{be}\;\langle \mu a.\, N, \overrightarrow{P}\rangle\;\texttt{in}\;M\\
&\quad =_{\mathrm{n}} \mu b.\, N\{[b](\texttt{box}\,\langle\overrightarrow{x}\rangle\;\texttt{be}\;\langle -, \overrightarrow{P}\rangle\;\texttt{in}\;M)/[a]-\}
\end{aligned}$$

where $\{C/[a]-\}$ means the substitution of $C\{M\}$ for $[a]M$, $C\{\mu a.\,[a,b]M\}$ for $[a,b]M$, and $C\{\mu a.\,[b,a]M\}$ for $[b,a]M$.

Proof. Straightforwardly. Note that $[\![M\{N/x\}]\!]_{\mathrm{n}} \equiv [\![M]\!]_{\mathrm{n}}\{[\![N]\!]_{\mathrm{n}}/x\}$ holds. □

The fact, that the $\lambda\mu\Box$-calculus satisfies box axioms of the $\lambda\Box$-calculus, means that the modality in the $\lambda\mu\Box$-calculus is a monoidal functor.

Remark 12. $\Box$ is neither symmetric nor strong in the $\lambda\mu\Box$-calculus even if $\Box$ is symmetric and strong in the target of the CPS transformation.

Though the $\lambda\mu\Box$-calculus does not have a diamond modality primitively, we can define a construct for a diamond modality. Let $\Diamond\tau$ be $\neg\Box\neg\tau$. Define syntax sugar by

$$\begin{aligned}
&\texttt{dia}\,\langle\overrightarrow{a}\rangle\;\texttt{be}\;\langle\overrightarrow{N}\rangle\;\texttt{in}\;M\\
&\qquad \equiv \lambda k.\, \overrightarrow{N}(\lambda\overrightarrow{x'}.\, k(\texttt{box}\,\langle\overrightarrow{x}\rangle\;\texttt{be}\;\langle\overrightarrow{x'}\rangle\;\texttt{in}\;\mu b.\,\overrightarrow{x}(\mu\overrightarrow{a}.\,[b]M)))
\end{aligned}$$

for terms M and $\overrightarrow{N}$ such that $\vdash M : \neg\tau \mid a_1 : \sigma_1, \ldots, a_n : \sigma_n$ and $\mathit{\Gamma} \vdash N_j : \neg\Diamond\sigma_j \mid \mathit{\Delta}$ hold. Then the judgment

$$\mathit{\Gamma} \vdash \texttt{dia}\,\langle a_1^{\sigma_1}, \ldots, a_n^{\sigma_n}\rangle\;\texttt{be}\;\langle N_1, \ldots, N_n\rangle\;\texttt{in}\;M : \neg\Diamond\tau \mid \mathit{\Delta}$$

is derivable. This $\texttt{dia}\,\langle\overrightarrow{a}\rangle\;\texttt{be}\;\langle\overrightarrow{N}\rangle\;\texttt{in}\;M$ is a dual form of $\texttt{box}\,\langle\overrightarrow{x}\rangle\;\texttt{be}\;\langle\overrightarrow{N}\rangle\;\texttt{in}\;M$ in the sense of Section 5. A formula $\Diamond(\tau_1 \vee \tau_2) \to \Diamond\tau_1 \vee \Diamond\tau_2$, which means distributivity of $\Diamond$ to $\vee$, is inherited by the term

$$\lambda x.\, \mu(a_1', a_2').\, (\texttt{dia}\,\langle a_1, a_2\rangle\;\texttt{be}\;\langle[a_1'], [a_2']\rangle\;\texttt{in}\;[[a_1], [a_2]])x$$

where we write just $[a]$ for $\lambda x.\,[a]x$. It is remarkable that the family of these terms is not a natural transformation. One can find more properties of $\Diamond$ through the duality.

In a similar way, we can consider another modality $\Box'\tau \equiv \neg\neg\Box\neg\neg\tau$ and

$$\begin{aligned}&\mathtt{box}'\,\langle\overrightarrow{x}\rangle\;\mathtt{be}\;\langle\overrightarrow{N}\rangle\;\mathtt{in}\;M\\ &\quad\equiv \mathtt{dia}\,\langle\overrightarrow{a}\rangle\;\mathtt{be}\;\langle\overrightarrow{N}\rangle\;\mathtt{in}\;\lambda k.\,[\overrightarrow{a}](\lambda\overrightarrow{x}.\,kM)\\ &\quad=_{\mathrm{n}} \lambda h.\,\overrightarrow{N}(\lambda\overrightarrow{y'}.\,h(\mathtt{box}\,\langle\overrightarrow{y}\rangle\;\mathtt{be}\;\langle\overrightarrow{y'}\rangle\;\mathtt{in}\;\overrightarrow{y}(\lambda\overrightarrow{x}.\,M)))\end{aligned}$$

for terms M and $\overrightarrow{N}$ such that $x_1 : \sigma_1, \ldots, x_n : \sigma_n \;\vdash\; M : \tau \;\mid\;$ and $\Gamma \;\vdash\; N_j : \Box'\sigma_j \;\mid\; \Delta$ hold. Because $\neg\neg\tau$ is not isomorphic to τ in general in the classical logic, $\Box'\tau$ is not isomorphic to $\Box\tau$ in the $\lambda\mu\Box$-calculus. However, $\Box'$ acts like $\Box$; the following is a formal description of this fact.

Theorem 13. *Define the transformation* $\overline{-}$ *on the* $\lambda\mu\Box$*-calculus by* $\overline{\Box\tau} \equiv \Box'\tau$ *and* $\overline{\mathtt{box}\,\langle\overrightarrow{x}\rangle\;\mathtt{be}\;\langle\overrightarrow{N}\rangle\;\mathtt{in}\;M} \equiv \mathtt{box}'\,\langle\overrightarrow{x}\rangle\;\mathtt{be}\;\langle\overrightarrow{\overline{N}}\rangle\;\mathtt{in}\;\overline{M}$*. For* $\lambda\mu\Box$*-terms* M *and* N*,* $M =_{\mathrm{n}} N$ *holds if and only if* $\overline{M} =_{\mathrm{n}} \overline{N}$ *holds.*

Proof. Consider the transformation on the $\lambda\Box$-calculus that sends $[\![M]\!]_{\mathrm{n}}$ to $[\![\overline{M}]\!]_{\mathrm{n}}$, *i.e.*, $\Box$ to $(\Box((- \to \mathtt{R}) \to \mathtt{R}) \to \mathtt{R}) \to \mathtt{R}$. The claim of the theorem is nothing less than this transformation preserves and reflects the equality. One can show the left-to-right implication by checking all the axioms. The right-to-left implication holds because the functor $(- \to \mathtt{R}) \to \mathtt{R}$ is injective in the $\lambda\Box$-calculus. $\Box$

4 Call-by-Value Calculus for Normal Modal Logic

In this section, we provide a call-by-value calculus which is dual to the call-by-name $\lambda\mu\Box$-calculus. A formal statement of the duality is given in the next section. The call-by-value calculus is an extension of Selinger's call-by-value version [30] of the $\lambda\mu$-calculus, and hence an extension of the λ_{c}-calculus [24].

Definition 14. *The call-by-value* $\lambda\mu\Box$*-calculus has the same syntax as the call-by-name* $\lambda\mu\Box$*-calculus. The equality of the call-by-value* $\lambda\mu\Box$*-calculus is defined by the transformation* $[\![-]\!]_{\mathrm{v}}$ *given in Table 5. We write* $M =_{\mathrm{v}} N$ *for* $[\![M]\!]_{\mathrm{v}} = [\![N]\!]_{\mathrm{v}}$ *when* M *and* N *have the same type.*

Since the call-by-value $\lambda\mu\Box$-calculus has the same syntax as the call-by-name, it is trivial that the call-by-value calculus Curry-Howard corresponds to **K**.

For an axiomatization of the call-by-value calculus, we need to define the set of values. Values V and evaluation contexts E are defined by

$$\begin{aligned}V &::= \alpha^\tau \mid x \mid \lambda x^\tau.\,M \mid \langle\,\rangle \mid \langle V, V\rangle \mid \pi_1 V \mid \pi_2 V\\ &\quad\mid [\lambda x^\tau.\,M, \lambda x^\tau.\,M] \mid [\lambda x^\tau.\,V, \lambda x^\tau.\,V]V \mid \iota_1 V \mid \iota_2 V\\ &\quad\mid \mathtt{box}\,\langle x^\tau, \ldots, x^\tau\rangle\;\mathtt{be}\;\langle V, \ldots, V\rangle\;\mathtt{in}\;M\\ E &::= - \mid EM \mid VE \mid \langle E, M\rangle \mid \langle V, E\rangle \mid \pi_1 E \mid \pi_2 E \mid [a]E \mid [a,a]E\\ &\quad\mid \mathtt{box}\,\langle x^\tau, \ldots, x^\tau\rangle\;\mathtt{be}\;\langle V, \ldots, V, E, M, \ldots, M\rangle\;\mathtt{in}\;M\end{aligned}$$

Table 5. CBV CPS transformation

$$
\begin{array}{rlrl}
p^\bullet &\equiv p & (\Box\tau)^\bullet &\equiv (\Box((\tau^\bullet \to \mathtt{R}) \to \mathtt{R}) \to \mathtt{R}) \to \mathtt{R} \\
(\neg\sigma)^\bullet &\equiv \sigma^\bullet \to \mathtt{R} & (\sigma \to \tau)^\bullet &\equiv (\sigma^\bullet \wedge (\tau^\bullet \to \mathtt{R})) \to \mathtt{R} \;\text{ if } \tau \not\equiv \bot \\
\top^\bullet &\equiv \top & (\tau_1 \wedge \tau_2)^\bullet &\equiv {\tau_1}^\bullet \wedge {\tau_2}^\bullet \\
\bot^\bullet &\equiv \bot & (\tau_1 \vee \tau_2)^\bullet &\equiv {\tau_1}^\bullet \vee {\tau_2}^\bullet
\end{array}
$$

$$
\begin{aligned}
[\![\alpha]\!]_{\mathrm{v}} &\equiv \lambda k.\, k\alpha \\
[\![x]\!]_{\mathrm{v}} &\equiv \lambda k.\, kx \\
[\![\lambda x.\, M]\!]_{\mathrm{v}} &\equiv \lambda k.\, k(\lambda x.\, [\![M]\!]_{\mathrm{v}}[\,]) \;\text{ if }\; M : \bot \\
&\equiv \lambda k.\, k(\lambda\langle x, h\rangle.\, [\![M]\!]_{\mathrm{v}} h) \;\text{ o.w.} \\
[\![MN]\!]_{\mathrm{v}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{v}}[\![N]\!]_{\mathrm{v}} \;\text{ if }\; MN : \bot \\
&\equiv \lambda k.\, [\![M]\!]_{\mathrm{v}}(\lambda y.\, [\![N]\!]_{\mathrm{v}}(\lambda z.\, y\langle z, k\rangle)) \;\text{ o.w.} \\
[\![\langle\,\rangle]\!]_{\mathrm{v}} &\equiv \langle\,\rangle \\
[\![\langle M_1, M_2\rangle]\!]_{\mathrm{v}} &\equiv \lambda k.\, [\![M_1]\!]_{\mathrm{v}}(\lambda y_1.\, [\![M_2]\!]_{\mathrm{v}}(\lambda y_2.\, k\langle y_1, y_2\rangle)) \\
[\![\pi_j M]\!]_{\mathrm{v}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{v}}(\lambda y.\, k(\pi_j y)) \\
[\![\mu a.\, M]\!]_{\mathrm{v}} &\equiv \lambda a.\, [\![M]\!]_{\mathrm{v}}[\,] \\
[\![[a]M]\!]_{\mathrm{v}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{v}} a \\
[\![\mu(a_1, a_2).\, M]\!]_{\mathrm{v}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{v}}\{\lambda y_1.\, k(\iota_1 y_1), \lambda y_2.\, k(\iota_2 y_2)/a_1, a_2\}[\,] \\
[\![[a_1, a_2]M]\!]_{\mathrm{v}} &\equiv \lambda k.\, [\![M]\!]_{\mathrm{v}}[a_1, a_2]
\end{aligned}
$$

$$
[\![\mathtt{box}\,\langle\vec{x}\rangle\;\mathtt{be}\;\langle\vec{N}\rangle\;\mathtt{in}\;M]\!]_{\mathrm{v}} \equiv
$$

$$
\lambda k.\, [\![\vec{N}]\!]_{\mathrm{v}}(\lambda\vec{g}.\, k(\lambda h.\, \vec{g}(\lambda\vec{f'}.\, h(\mathtt{box}\,\langle\vec{f}\rangle\;\mathtt{be}\;\langle\vec{f'}\rangle\;\mathtt{in}\;\lambda l.\, \vec{f}(\lambda\vec{x}.\, [\![M]\!]_{\mathrm{v}} l)))))
$$

$$
\frac{x_1 : \sigma_1, \ldots, x_n : \sigma_n \;\vdash\; M : \tau \;\mid\; a_1 : \tau_1, \ldots, a_m : \tau_m}{x_1 : {\sigma_1}^\bullet, \ldots, x_n : {\sigma_n}^\bullet, a_1 : {\tau_1}^\bullet \to \mathtt{R}, \ldots, a_m : {\tau_m}^\bullet \to \mathtt{R} \;\vdash\; [\![M]\!]_{\mathrm{v}} : (\tau^\bullet \to \mathtt{R}) \to \mathtt{R}}
$$

and we use also W as a meta-variable for values. In addition, we use the syntax sugar

$$
\begin{aligned}
\mathtt{let}\, x\;\mathtt{be}\; N\;\mathtt{in}\; M &\equiv (\lambda x.\, M)N \\
\mathtt{let}\;\vec{x}\;\mathtt{be}\;\vec{N}\;\mathtt{in}\; M &\equiv \mathtt{let}\, x_1\;\mathtt{be}\; N_1\;\mathtt{in}\cdots\mathtt{let}\, x_n\;\mathtt{be}\; N_n\;\mathtt{in}\; M
\end{aligned}
$$

as usual. Syntax sugar about disjunctions is used in the definition of values, but it is possible to introduce $[M_1, M_2]$ and $\iota_j M$ as primitive syntax instead of $\mu(a_1, a_2).\, M$ and $[a_1, a_2]M$ in the call-by-value calculus as noted in [31].

In our definition, there is a value that has a redex exterior to abstractions, but it is not serious because we are not focusing on reductions. Our notion of values is based on semantical effect-freeness: a value is interpreted to a form $\lambda k.\, kV$. There is room for improvement if we consider not an equality but a reduction system or other semantics.

Proposition 15. *The following equations hold in the call-by-value $\lambda\mu\Box$-calculus.*

$$
\begin{aligned}
&E\{M\} =_{\mathrm{v}} \mathtt{let}\,x\,\mathtt{be}\,M\ \mathtt{in}\ E\{x\} && \text{if } x \notin \mathtt{FV}(E)\\
&\mathtt{box}\,\langle x\rangle\ \mathtt{be}\ \langle M\rangle\ \mathtt{in}\ x =_{\mathrm{v}} M\\
&\mathtt{box}\,\langle \overrightarrow{z}, x\rangle\ \mathtt{be}\ \langle \overrightarrow{P}, \mathtt{box}\,\langle \overrightarrow{y}\rangle\ \mathtt{be}\ \langle \overrightarrow{L}\rangle\ \mathtt{in}\ N\rangle\ \mathtt{in}\ M\\
&\quad =_{\mathrm{v}} \mathtt{box}\,\langle \overrightarrow{z}, \overrightarrow{y}\rangle\ \mathtt{be}\ \langle \overrightarrow{P}, \overrightarrow{L}\rangle\ \mathtt{in}\ \mathtt{let}\,x\,\mathtt{be}\,N\ \mathtt{in}\ M\\
&\mathtt{box}\,\langle \overrightarrow{w}, x, \overrightarrow{z}\rangle\ \mathtt{be}\ \langle \overrightarrow{W}, \mathtt{box}\,\langle \overrightarrow{y}\rangle\ \mathtt{be}\ \langle \overrightarrow{N}\rangle\ \mathtt{in}\ V, \overrightarrow{P}\rangle\ \mathtt{in}\ M\\
&\quad =_{\mathrm{v}} \mathtt{box}\,\langle \overrightarrow{w}, \overrightarrow{y}, \overrightarrow{z}\rangle\ \mathtt{be}\ \langle \overrightarrow{W}, \overrightarrow{N}, \overrightarrow{P}\rangle\ \mathtt{in}\ M\{V/x\} && \text{if } |\overrightarrow{w}| = |\overrightarrow{W}|
\end{aligned}
$$

where V and W_j are values.

Proof. First we show the following fact by induction: for any value V, there is a term V' such that $[\![V]\!]_{\mathrm{v}} = \lambda k.\, kV'$. This fact enables us to show $[\![E\{M\}]\!]_{\mathrm{v}} = \lambda k.\, [\![M]\!]_{\mathrm{v}}(\lambda x.\, [\![E\{x\}]\!]_{\mathrm{v}}k)$ by induction. Also the last equation is derived from the fact. Other equations are proved straightforwardly. $\Box$

Commutativity between μ abstractions and boxed applications is derived from the first equation:

$$
\begin{aligned}
&\mathtt{box}\,\langle \overrightarrow{x}\rangle\ \mathtt{be}\ \langle \overrightarrow{W}, \mu a.\, N, \overrightarrow{P}\rangle\ \mathtt{in}\ M\\
&\quad =_{\mathrm{v}} \mathtt{let}\,y\,\mathtt{be}\,\mu a.\, N\ \mathtt{in}\ \mathtt{box}\,\langle \overrightarrow{x}\rangle\ \mathtt{be}\ \langle \overrightarrow{W}, y, \overrightarrow{P}\rangle\ \mathtt{in}\ M\\
&\quad =_{\mathrm{v}} \mu b.\, N\{[b](\mathtt{let}\,y\,\mathtt{be} - \mathtt{in}\ \mathtt{box}\,\langle \overrightarrow{x}\rangle\ \mathtt{be}\ \langle \overrightarrow{W}, y, \overrightarrow{P}\rangle\ \mathtt{in}\ M)/[a]-\}\\
&\quad =_{\mathrm{v}} \mu b.\, N\{[b](\mathtt{box}\,\langle \overrightarrow{x}\rangle\ \mathtt{be}\ \langle \overrightarrow{W}, -, \overrightarrow{P}\rangle\ \mathtt{in}\ M)/[a]-\}
\end{aligned}
$$

where the last two lines hold by the ordinary call-by-value equality. Unlike the call-by-name case, Proposition 15 means that $\Box$ in the call-by-value calculus is monoidal only on values.

We define the diamond structure $\Diamond\tau$ and $\mathtt{dia}\,\langle \overrightarrow{a}\rangle \mathtt{be} \langle \overrightarrow{N}\rangle\ \mathtt{in}\ M$ in the call-by-value $\lambda\mu\Box$-calculus as just the same syntax sugar as in the call-by-name. Such syntax is used for the duality in the next section.

Remark 16. For the duality, we adopt a complex transformation as semantics: $[\![-]\!]_{\mathrm{v}}$ is defined in order that

$$(\Box\tau)^{\bullet} \equiv (\Box((\tau^{\bullet} \to \mathtt{R}) \to \mathtt{R}) \to \mathtt{R}) \to \mathtt{R}$$

holds. If we ignore the duality, we can reduce the transformation such that

$$(\Box\tau)^{\bullet} \equiv \Box((\tau^{\bullet} \to \mathtt{R}) \to \mathtt{R})$$

holds. It can be proved that this simpler translation gives the same equality as the original one.

Table 6. Transformation from CBV to CBN

$$\lceil p \rceil \equiv p \qquad \lceil \neg\sigma \rceil \equiv \neg\lceil \sigma \rceil$$

$$\lceil \sigma \to \tau \rceil \equiv \neg(\lceil \sigma \rceil \vee \neg\lceil \tau \rceil) \text{ if } \tau \not\equiv \bot$$

$$\lceil \top \rceil \equiv \bot \qquad \lceil \tau_1 \wedge \tau_2 \rceil \equiv \lceil \tau_1 \rceil \vee \lceil \tau_2 \rceil$$

$$\lceil \bot \rceil \equiv \top \qquad \lceil \tau_1 \vee \tau_2 \rceil \equiv \lceil \tau_1 \rceil \wedge \lceil \tau_2 \rceil$$

$$\lceil \Box\tau \rceil \equiv \Diamond\lceil \tau \rceil$$

$$\lceil \texttt{box}\,\langle \overrightarrow{x} \rangle\ \texttt{be}\ \langle \overrightarrow{N} \rangle\ \texttt{in}\ M \rceil \equiv \texttt{dia}\,\langle \overrightarrow{x} \rangle\ \texttt{be}\ \langle \lceil \overrightarrow{N} \rceil \rangle\ \texttt{in}\ \lceil M \rceil$$

$$\frac{x_1 : \sigma_1, \ldots, x_n : \sigma_n \ \vdash\ M : \tau \ \mid\ a_1 : \tau_1, \ldots, a_m : \tau_m}{a_1 : \lceil \tau_1 \rceil, \ldots, a_m : \lceil \tau_m \rceil \ \vdash\ \lceil M \rceil : \neg\lceil \tau \rceil \ \mid\ x_1 : \lceil \sigma_1 \rceil, \ldots, x_n : \lceil \sigma_n \rceil}$$

Table 7. Transformation from CBN to CBV

$$\lfloor p \rfloor \equiv p \qquad \lfloor \neg\sigma \rfloor \equiv \neg\lfloor \sigma \rfloor$$

$$\lfloor \sigma \to \tau \rfloor \equiv \neg\lfloor \sigma \rfloor \wedge \lfloor \tau \rfloor \text{ if } \tau \not\equiv \bot$$

$$\lfloor \top \rfloor \equiv \bot \qquad \lfloor \tau_1 \wedge \tau_2 \rfloor \equiv \lfloor \tau_1 \rfloor \vee \lfloor \tau_2 \rfloor$$

$$\lfloor \bot \rfloor \equiv \top \qquad \lfloor \tau_1 \vee \tau_2 \rfloor \equiv \lfloor \tau_1 \rfloor \wedge \lfloor \tau_2 \rfloor$$

$$\lfloor \Diamond\tau \rfloor \equiv \Box\lfloor \tau \rfloor$$

$$\lfloor \texttt{dia}\,\langle \overrightarrow{x} \rangle\ \texttt{be}\ \langle \overrightarrow{N} \rangle\ \texttt{in}\ M \rfloor \equiv \lambda k.\, \lfloor \overrightarrow{N} \rfloor(\lambda \overrightarrow{x'}.\, k(\texttt{box}\,\langle \overrightarrow{x} \rangle\ \texttt{be}\ \langle \overrightarrow{x'} \rangle\ \texttt{in}\ \mu a.\, \lfloor M \rfloor(\lambda y.\, [a]y)))$$

$$\frac{a_1 : \tau_1, \ldots, a_m : \tau_m \ \vdash\ M : \sigma \ \mid\ x_1 : \sigma_1, \ldots, x_n : \sigma_n}{x_1 : \lfloor \sigma_1 \rfloor, \ldots, x_n : \lfloor \sigma_n \rfloor \ \vdash\ \lfloor M \rfloor : \neg\lfloor \sigma \rfloor \ \mid\ a_1 : \lfloor \tau_1 \rfloor, \ldots, a_m : \lfloor \tau_m \rfloor}$$

5 Duality between Call-by-Name and Call-by-Value

It is known that there exists a duality between call-by-name and call-by-value in languages with control operators, *e.g.*, [10] and [30]. In this section, we observe such duality on the $\lambda\mu\Box$-calculus. Since our calculi are extensions of Selinger's $\lambda\mu$-calculi, we show the duality along the line of [30].

For readability of the duality, we use meta-variables a and x for variables and control variables of the call-by-name $\lambda\mu\Box$-calculus, respectively.

Table 6 gives the transformation from the call-by-value to the call-by-name $\lambda\mu\Box$-calculus. Other cases than the box case are omitted in the table because they are essentially the same as Selinger's [30]. It is shown that the call-by-value CPS transformation coincides with the call-by-name one via this transformation.

Theorem 17. *For any type τ and any term M of the call-by-value $\lambda\mu\Box$-calculus, $\tau^\bullet \equiv \lceil \tau \rceil^\circ$ and $[\![M]\!]_{\mathrm{v}} = [\![\lceil M \rceil]\!]_{\mathrm{n}}$ hold.*

Proof. By induction. □

On the other hand, a transformation from the call-by-name to the call-by-value can not be defined totally. We just define the transformation from the $\Diamond$

fragment of the call-by-name $\lambda\mu\Box$-calculus to the call-by-value $\lambda\mu\Box$-calculus by Table 7. Since the type of $[\![M]\!]_{\mathrm{n}}$ does not match the type of $[\![\lfloor M\rfloor]\!]_{\mathrm{v}}$, the dual of the previous theorem is the following.

Theorem 18. *For any type τ and any term M of the $\Diamond$ fragment of the call-by-name $\lambda\mu\Box$-calculus, $\tau^\circ \equiv \lfloor\tau\rfloor^\bullet$ and $[\![M]\!]_{\mathrm{n}} = \lambda x.\, [\![\lfloor M\rfloor]\!]_{\mathrm{v}}(\lambda k.\, kx)$ hold.*

Proof. By induction. $\Box$

It follows from Theorem 17 and 18 that the call-by-value $\lambda\mu\Box$-calculus and the $\Diamond$ fragment of the call-by-name $\lambda\mu\Box$-calculus are in bijective correspondence in some sense. Moreover,

$$\lfloor \mathtt{box}'\,\langle\overrightarrow{a}\rangle\ \mathtt{be}\ \langle\overrightarrow{N}\rangle\ \mathtt{in}\ M\rfloor =_{\mathrm{v}} \mathtt{dia}\,\langle\overrightarrow{a}\rangle\ \mathtt{be}\ \langle\lfloor\overrightarrow{N}\rfloor\rangle\ \mathtt{in}\ \lfloor M\rfloor$$

holds. Hence, there exists a bijective correspondence between the $\Box'$ fragment of the call-by-name $\lambda\mu\Box$-calculus and the $\Diamond$ fragment of the call-by-value $\lambda\mu\Box$-calculus. By Theorem 13 in Section 3, we can conclude that the call-by-name $\lambda\mu\Box$-calculus and the $\Diamond$ fragment of the call-by-value $\lambda\mu\Box$-calculus are in bijective correspondence.

6 Extensions

We add type-indexed families of constants $\{\varepsilon_\sigma : \Box\sigma \to \sigma\}$ and $\{\delta_\sigma : \Box\sigma \to \Box\Box\sigma\}$ with the axioms

$$\begin{aligned}
&\varepsilon(\mathtt{box}\,\langle\overrightarrow{x}\rangle\ \mathtt{be}\ \langle\overrightarrow{N}\rangle\ \mathtt{in}\ M) = M\{\varepsilon\overrightarrow{N}/\overrightarrow{x}\}\\
&\delta(\mathtt{box}\,\langle\overrightarrow{x}\rangle\ \mathtt{be}\ \langle\overrightarrow{N}\rangle\ \mathtt{in}\ M) = \mathtt{box}\,\langle\overrightarrow{y}\rangle\ \mathtt{be}\ \langle\delta\overrightarrow{N}\rangle\ \mathtt{in}\ \mathtt{box}\,\langle\overrightarrow{x}\rangle\ \mathtt{be}\ \langle\overrightarrow{y}\rangle\ \mathtt{in}\ M\\
&\delta(\delta M) = \mathtt{box}\,\langle x\rangle\ \mathtt{be}\ \langle\delta M\rangle\ \mathtt{in}\ \delta x\\
&\varepsilon(\delta M) = \mathtt{box}\,\langle x\rangle\ \mathtt{be}\ \langle\delta M\rangle\ \mathtt{in}\ \varepsilon x = M
\end{aligned}$$

to the $\lambda\Box$-calculus. A model of this calculus is a bicartesian closed category with a monoidal comonad, that is, a model of the box fragment of **IS4**. Also to the $\lambda\mu\Box$-calculus, we add families $\{\varepsilon_\sigma : \Box\sigma \to \sigma\}$ and $\{\delta_\sigma : \Box\sigma \to \Box\Box\sigma\}$. Then, it is obvious that this calculus corresponds to **S4** with respect to provability. The semantics are given by

$$\begin{aligned}
&[\![\varepsilon]\!]_{\mathrm{n}} \equiv \lambda\langle x,k\rangle.\, x(\lambda y.\, \varepsilon yk)\\
&[\![\delta]\!]_{\mathrm{n}} \equiv \lambda\langle x,k\rangle.\, x(\lambda y.\, k(\mathtt{box}\,\langle z\rangle\ \mathtt{be}\ \langle\delta y\rangle\ \mathtt{in}\ \lambda h.\, hz))\\
&[\![\varepsilon]\!]_{\mathrm{v}} \equiv \lambda k.\, k(\lambda\langle x,h\rangle.\, x(\lambda y.\, \varepsilon yh))\\
&[\![\delta]\!]_{\mathrm{v}} \equiv \lambda k.\, k(\lambda\langle x,h\rangle.\, h(\lambda l.\, x(\lambda y.\, l(\mathtt{box}\,\langle z\rangle\ \mathtt{be}\ \langle\delta y\rangle\ \mathtt{in}\ \lambda m.\, m(\lambda n.\, nz))))).
\end{aligned}$$

Unfortunately, $\Box$ is not a comonad in the call-by-name calculus because

$$\varepsilon(\mathtt{box}\,\langle x\rangle\ \mathtt{be}\ \langle N\rangle\ \mathtt{in}\ M) \neq_{\mathrm{n}} M\{\varepsilon N/x\}$$

in general. On the other hand, in the call-by-value calculus, the equations

$$\begin{aligned}
&\varepsilon(\texttt{box}\,\langle x\rangle\,\texttt{be}\,\langle N\rangle\,\texttt{in}\,M) =_{\mathrm{v}} \texttt{let}\,x\,\texttt{be}\,\varepsilon N\,\texttt{in}\,M\\
&\delta(\texttt{box}\,\langle x\rangle\,\texttt{be}\,\langle N\rangle\,\texttt{in}\,M) =_{\mathrm{v}} \texttt{box}\,\langle y\rangle\,\texttt{be}\,\langle \delta N\rangle\,\texttt{in}\,\texttt{box}\,\langle x\rangle\,\texttt{be}\,\langle y\rangle\,\texttt{in}\,M\\
&\delta(\delta M) =_{\mathrm{v}} \texttt{box}\,\langle x\rangle\,\texttt{be}\,\langle \delta M\rangle\,\texttt{in}\,\delta x\\
&\varepsilon(\delta M) =_{\mathrm{v}} \texttt{box}\,\langle x\rangle\,\texttt{be}\,\langle \delta M\rangle\,\texttt{in}\,\varepsilon x =_{\mathrm{v}} M
\end{aligned}$$

hold, and hence $\Box$ is a comonad (but not a monoidal comonad). Through the duality, one can conclude that $\Diamond$ is a monad in the call-by-name calculus.

In [6], Bierman and de Paiva propose a monad as a model of $\Diamond$ in **IS4**. Our semantics matches their observation. An **S4** extension of the dual calculus [34] along the line of dual context calculi (*e.g.*, [2]) is provided in [32] by Shan. Since the $\lambda\mu$-calculus has a bijective correspondence to the dual calculus, the $\lambda\mu\Box$-calulus remains to be formalized in the dual calculus and to be compared with Shan's calculus.

References

[1] T. Abe. Completeness of modal proofs in first-order predicate logic. *Computer Software, JSSST Journal*, to appear.

[2] A. Barber. Dual intuitionistic linear logic. Technical report, LFCS, University of Edinburgh, 1996.

[3] H. P. Barendregt. Lambda calculi with types. In S. Abramski, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, pages 117–309. Oxford University Press, 1992.

[4] G. Bellin, V. C. V. de Paiva, and E. Ritter. Extended Curry-Howard correspondence for a basic constructive modal logic. In *Proceedings of Methods for Modalities*, 2001.

[5] G. M. Bierman. What is a categorical model of intuitionistic linear logic. In *Typed Lambda Calculi and Applications*, volume 902 of *LNCS*, pages 78–93. Springer-Verlag, 1995.

[6] G. M. Bierman and V. C. V. de Paiva. On an intuitionistic modal logic. *Studia Logica*, 65(3):383–416, 2000.

[7] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, 2001.

[8] R. Davies and F. Pfenning. A modal analysis of staged computation. *Journal of the ACM*, 48(3):555–604, 2001.

[9] P. de Groote. A cps-translation of the $\lambda\mu$-calculus. In *Colloquium on Trees in Algebra and Programming*, volume 787 of *LNCS*, pages 85–99. Springer-Verlag, 1994.

[10] A. Filinski. Declarative continuations and categorical duality. Master's thesis, Computer Science Department, University of Copenhagen, 1989.

[11] M. Fischer. Lambda calculus schemata. In *Proving Assertions about Programs*, pages 104–109. ACM Press, 1972.

[12] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50(1):1–102, 1987.

[13] T. G. Griffin. A formulae-as-types notion of control. In *Principles of Programming Languages*, pages 47–58. ACM Press, 1990.

[14] M. Hofmann and T. Streicher. Continuation models are universal for $\lambda\mu$-calculus. In *Logic in Computer Science*, pages 387–397. IEEE Computer Society, 1997.

[15] W. A. Howard. The formulae-as-types notion of construction. In *Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 479–490. Academic Press, 1980.

[16] Y. Kakutani. Duality between call-by-name recursion and call-by-value iteration. In *Computer Science Logic*, volume 2471 of *LNCS*, pages 506–521. Springer-Verlag, 2002.

[17] Y. Kakutani. Calculi for intuitionistic normal modal logic. In *Proceedings of Programming and Programming Languages*, 2007.

[18] S. Kripke. Semantic analysis of modal logic I, normal propositional logic. *Zeitschrift für Mathemathische Logik und Grundlagen der Mathematik*, 9:67–96, 1963.

[19] J. Lambek and P. J. Scott. *Introduction to Higher-Order Categorical Logic*. Cambridge University Press, 1986.

[20] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, 2nd edition, 1997.

[21] M. E. Maietti, P. Maneggia, V. C. V. de Paiva, and E. Ritter. Relating categorical semantics for intuitionistic linear logic. *Applied Categorical Structures*, 13(1):1–36, 2005.

[22] S. Martini and A. Masini. A computational interpretation of modal proofs. In *Proof Theory of Modal Logics*, pages 213–241. Kluwer, 1996.

[23] K. Miyamoto and A. Igarashi. A modal foundation for secure information flow. In *Proceedings of Foundations of Computer Security*, 2004.

[24] E. Moggi. Computational lambda-calculus and monads. In *Logic in Computer Science*, pages 14–23. IEEE Computer Society, 1989.

[25] C.-H. L. Ong and C. A. Stewart. A Curry-Howard foundation for functional computation with control. In *Principle of Programming Languages*, pages 215–227. ACM Press, 1997.

[26] M. Parigot. $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction. In *Logic Programming and Automated Reasoning*, volume 624 of *LNCS*, pages 190–201. Springer-Verlag, 1992.

[27] G. D. Plotkin. Call-by-name, call-by-value and the lambda calculus. *Theoretical Computer Science*, 1(2):125–159, 1975.

[28] D. Pym and E. Ritter. On the semantics of classical disjunction. *Journal of Pure and Applied Algebra*, 159(2,3):315–338, 2001.

[29] R. A. G. Seely. Linear logic, $*$-autonomous categories and cofree coalgebras. In *Categories in Computer Science and Logic*, volume 92 of *Contemporary Mathematics*, pages 371–389. AMS, 1989.

[30] P. Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science*, 11(2):207–260, 2001.

[31] P. Selinger. Some remarks on control categories. Manuscript, 2003.

[32] C.-C. Shan. A computastional interpretation of classical S4 modality. In *Proceedings of Intuitionistic Modal Logics and Applications*, 2005.

[33] A. K. Simpson. *The Proof Theory and Semantics of Intuitionistic Modal Logics*. PhD thesis, University of Edinburgh, 1993.

[34] P. Wadler. Call-by-value is dual to call-by-name. In *International Conference on Functional Programming*, pages 189–201. ACM Press, 2003.

[35] D. Wijesekera. Constructive modal logic I. *Annals of Pure and Applied Logic*, 50:271–301, 1990.