GRAPHICAL CLASSIFICATION OF ENTANGLED QUTRITS

by

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ABSTRACT

Quantum entanglement is a multipartite quantum state from which each quantum cannot be separated. It plays a fundamental role in many applications of quantum information, such as quantum teleportation.

Stochastic local quantum operations and classical communication (SLOCC) cannot essentially change quantum entanglement without destroying it. Therefore, classification of quantum states into groups in which quantum states can be converted to each other by SLOCC implies classification by entanglement of quantum states. However, classification by SLOCC in general requires complex calculation which is difficult to complete. Graphical representation can solve this problem. Classification by SLOCC of quantum bits, which is called qubits, has been expressed graphically by connection between tripartite entangled qubits and commutative Frobenius algebras (CFAs) in monoidal categories.

In this paper, we extended this method to qutrits, i.e., systems that have three bases. We examined correspondence between CFAs and tripartite entangled qutrits. As a result, using symmetry which CFAs require, we found that only three classes correspond to CFAs. We represented qutrits graphically by connection to CFAs, and derived equations which characterise the three classes. Moreover, we showed that any qutrit can be represented as a composite of three graphs which correspond to the three classes.

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Contents

Chapter 1

Introduction

For difficulty of miniaturisation of electronic component, it is said that the limit of Moore's law is coming [1]. One of strong solutions for that problem is the development of quantum computers. This uses quantum effect, which is contradicts the intuition in classical computing. In quantum computing, sending quantum information may be needed, but, quantum information cannot replicate [2]. To send quantum information to another person, foundational methods of quantum information theory, such as quantum teleportation [3], use quantum entanglement, from which we cannot separate a state of each side. Quantum entanglement is non-local property of quantum states, so entanglement does not increase by stochastic local quantum operations and classical communication (SLOCC). The entangled states are divided into some classes by the SLOCC-equivalence that is a relation between states which can be converted to each other by SLOCC. Recently, SLOCC-equivalent classes have been studied in [4][5][6][7][8][9]. There are too many SLOCC-equivalent classes to distinguish which classes a state belongs to. A hard calculation on complex numbers is usually needed to solve the problem. Graphical representation is easier way to solve the question. We focused, in this paper, on graphical representation of qutrits, which are systems having three dimensional state spaces. Qutrits have some superior points than quantum bits called qubits, which is studied well [10][11][12].

It has been known that arrows of categories can be expressed graphically [13]. Moreover, recently, quantum protocols and computing have been interpreted on monoidal categories, which are categories with a tensor product. Abramsky and Coecke gave quantum axioms and interpretation of quantum protocols on a kind of monoidal category called a biproduct dagger compact closed category [14]. Selinger gave categorical semantics for a quantum programming language QPL by the category [15][16]. Using graphical representation of arrows on monoidal categories and connections between quantum information and categories, qutrits also can be expressed graphically.

To express qutrits graphically, we adopted the method which is extension of the way in [17]. It was shown in [17] that highly entangled and highly symmetric qubits correspond to commutative Frobenius algebras (CFAs) on monoidal categories. Using the corresponding, they gave graphical representation of qubits. This representation reflects how entangled states are. In this paper, we used the way for qutrits. Using symmetry, we found that infinite classes do not correspond to CFAs and only three classes correspond to CFAs. We characterised each class corresponding to CFA by the equations on graphs. These classification implies the algebraic structure of some kinds of qutrits. As a result, we got three kinds of graph. Finally, we showed that any qutrits can be expressed graphically using the three kinds of graph and single qutrits.

In chapter 2, we show physical preparation. We explain some axioms, definitions, and concepts such as qutrits, quantum entanglement, and SLOCC classes. In chapter 3, we show mathematical preparation. In the chapter, we explain the definition of a monoidal category and a CFA. Moreover, we show the graphical representation, some theorems, and the case of qubits which are shown in [17]. In chapter 4, we show the result of our study. We classify SLOCC classes to non-maximal or non-symmetric or Frobenius or the rest classes, and show that there is only three Frobenius classes. Moreover, we define an ISCFA, a kind of CFA, and prove the uniqueness of the correspondence between Frobenius classes and three CFAs, an SCFA, an ACFA, and an ISCFA. Finally, we show the way to construct any qutrit.

Chapter 2

Quantum Entanglement

In this chapter, we show some axioms and define some physical concepts, such as qubits, qutrits, and quantum entanglement. Firstly, we explain behaviour of single system and define some relational concepts. Secondly, we focus on a multiple system and define quantum entanglement. Finally, we show SLOCC classes of tripartite qutrits, which is main subject of this study.

2.1 Qutrit

First, we show the behaviour of single system. Systems correspond to vectors.

Axiom 2.1. *A state of any isolated system can be described completely by a unit vector of a complex Hilbert space. The vector and the space is called a state vector and a state space of the system respectively.*

Because of this axiom, we identify a system with a state vector of the system. Moreover, we use Bra-Ket notation to indicate a state vector and its conjugate transpose.

Definition 2.2 (Bra, Ket)**.** *A state vector is written as* $|\psi\rangle$ *with an appropriate label* ψ*. A conjugate transpose of the vector is expressed as* $\langle\psi|$*. Moreover, an inner product of* $|\psi\rangle$ *and* $|\phi\rangle$ *are expressed as* $\langle\psi|\phi\rangle$*.*

This notation is called Bra-Ket notation.

Our main interest is restricted to special systems, i.e., a qubit and a qutrit.

Definition 2.3 (Qubit, Qutrit)**.** *A system which has a two dimensional Hilbert space as state space is called a qubit. A system which has a three dimensional Hilbert space as state space is called a qutrit.*

A qutrit is expressed by a vector in $\mathbb{C}^3$. Let $|0\rangle$, $|1\rangle$, and $|2\rangle$ be a canonical basis of $\mathbb{C}^3$, then any qutrit $|\psi\rangle$ is written as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \tag{2.1}$$

with complex numbers α, β, and γ such that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$. Like this, any qubit is expressed as linear combination of a canonical basis $\{|0\rangle, |1\rangle\}$of $\mathbb{C}^2$. In this paper, we use the canonical basis to express any qutrit and any function as matrix. For example, above $|\psi\rangle$ is expressed as

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{2.2}$$

Similarly, $\langle\psi|$ is written as

$$\langle\psi| = \begin{pmatrix} \alpha^* & \beta^* & \gamma^* \end{pmatrix} \tag{2.3}$$

α^* is a complex conjugate of α. Notice that $|\psi\rangle$ and $\langle\psi|$ can be regarded as a linear function from $\mathbb{C}$ to $\mathbb{C}^3$ and a linear function from $\mathbb{C}^3$ to $\mathbb{C}$ respectively.

The Above axiom and definitions are not strange. However some behaviour of system is counterintuitive. One of this behaviour appear when a system is measured. The behaviour follows an axiom below.

Axiom 2.4. *Let $|\psi\rangle$ be a state vector of a system and $\{|e_j\rangle\}_{j\in J}$ be an orthonormal basis of a state space of the system with a set J. Assume $|\psi\rangle$ is expressed as*

$$|\psi\rangle = \sum_{j\in J} \lambda_j |e_j\rangle \tag{2.4}$$

with complex numbers $\{\lambda_j\}_{j\in J}$. When the system is observed, the result state of the observation is $|e_k\rangle$ with probability $|\lambda_k|^2$. After $|e_k\rangle$ is observed, the state of the system is changed to $|e_k\rangle$. $\{|e_j\rangle\}_{j\in J}$ is called a measurement basis.

Consider $|\psi\rangle$ in (2.2), for example. When this system is observed with the canonical basis, the result is $|0\rangle$ with probability $|\alpha|^2$, $|1\rangle$ with probability $|\beta|^2$, and $|2\rangle$ with probability $|\gamma|^2$. Usually, observation is done with specification of a measurement basis. However, in this paper we use a canonical basis to measure any qutrit, so we omit specification of basis.

2.2 Quantum Entanglement

In the above section, we explained the strange behaviour of single system. Other peculiarity appears in a multiple system. We explain an axiom about a state vector of a multiple system.

Axiom 2.5. *Let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be Hilbert spaces, $|\psi_1\rangle, \dots, |\psi_n\rangle$ be unit vectors, and $Q_1, \dots, Q_n$ be systems. If H_i is a state space of Q_i for any $i \in \{1, \dots, n\}$, then the state space of a system Q which is composed by $Q_1, \dots, Q_n$ is $\bigotimes_{i=1}^n \mathcal{H}_i$. Moreover, if $|\psi_i\rangle$ is a state vector of Q_i for any $i \in \{1, \dots, n\}$, then a state vector of Q is $\bigotimes_{i=1}^n |\psi_i\rangle$.*

Notice that the axioms does not require that every state spaces are same.

We give a simple notation of a state vector of a multiple system.

$$|\psi_1\psi_2\cdots\psi_n\rangle \equiv |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle \tag{2.5}$$

We index qutrits by left-to-right order. For example, we say a ith qutrit of $|\psi_1\psi_2\cdots\psi_n\rangle$ to indicate $|\psi_i\rangle$.

Consider the axiom more closely. Bipartite qutrits can be written as a unit vector $|\psi\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$, so it can be written as

$$|\psi\rangle = \sum_{i,j\in\{0,1,2\}} \lambda_{ij}|ij\rangle \tag{2.6}$$

with complex number λ_{ij} such that $\sum_{i,j\in\{0,1,2\}} |\lambda_{ij}|^2 = 1$. If the state vector of the first qutrit is $|0\rangle$ and one of the second qutrit is $|1\rangle$, then $|\psi\rangle = |01\rangle$. In this example, λ_{ij} is zero except $(i,j) = (0,1)$. Assume $\lambda_{00} = \lambda_{11} = \frac{1}{\sqrt{2}}$ and other λ_{ij} is zero. In this case, $|\psi\rangle$ is

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{2.7}$$

This is a vector state of bipartite qutrits. However, there are no unit vectors $|\phi\rangle$ and $|\varphi\rangle$ such that $|\psi\rangle = |\phi\rangle \otimes |\varphi\rangle$. We call states like this entanglement.

Definition 2.6 (Entanglement, Separable)**.** *When a state vector of a multiple system can be written as a tensor product of state vectors of each system, the state is separable. When it cannot so, the state is entangled. An entangled state is called (quantum) entanglement.*

When $|\psi\rangle$ in (2.7) is observed, according to Axiom 2.4, $|00\rangle$ is observed in half probability and $|11\rangle$ is observed in the rest probability. We can measure only one qutrit of this bipartite qutrits. When the first qutrit is observed, the result is $|0\rangle$ or $|1\rangle$ with the same probability. When $|0\rangle$ is observed, the state of the second qutrit is $|0\rangle$. It means that we can know the state of the second qutrit without any operation to the second qutrit. This strange characteristic of a multiple system is used in protocols in quantum information such as quantum teleportation.

2.3 SLOCC

Whether the state is entanglement or not is an important question. Another important question is how entangled the state is. For example, we consider the following two tripartite qubits. In the following, we omit the normalised factor of states.

$$|\mathrm{GHZ}\rangle := |000\rangle + |111\rangle \tag{2.8}$$

and

$$|\mathrm{W}\rangle := |001\rangle + |010\rangle + |100\rangle \tag{2.9}$$

The states are called GHZ state and W state respectively. Let observe the first qubit of each tripartite qubit with a canonical basis of $\mathbb{C}^2$. After GHZ state is observed, the changed state is $|000\rangle$ or $|111\rangle$. Both states are separable. However, after W state is observed, the changed state is $|01\rangle+|10\rangle$ or $|00\rangle$. $|00\rangle$ is separable, but $|01\rangle + |10\rangle$ is entangled. GHZ state and W state is both entangled. How entangled the state is differ from each other. We need to classify systems by their entanglement.

Entanglement is a property of a multiple system, so local operations to each system cannot essentially change entanglement without destroying it. Classical communication does not change the property. Therefore, quantum local operations and classical communication (LOCC) does not essentially change entanglement of systems without destroying it. If a state $|\psi\rangle$ can be converted to $|\phi\rangle$ by LOCC with non-zero probability, it is said that $|\psi\rangle$ can be converted to $|\phi\rangle$ by stochastic local quantum operations and classical communication (SLOCC). Using SLOCC, we can define a classification of entanglement.

Definition 2.7 (SLOCC-equivalent)**.** *If states $|\psi\rangle$ and $|\phi\rangle$ can be converted to each other by SLOCC, then $|\psi\rangle$ and $|\phi\rangle$ are SLOCC-equivalent.*

Moreover, by SLOCC-equivalent, SLOCC-maximal can be defined.

Definition 2.8 (SLOCC-maximal)**.** *Let $|\psi\rangle$ be a state. If any $|\phi\rangle$ which can be converted to $|\psi\rangle$ by SLOCC is SLOCC-equivalent to $|\psi\rangle$, then $|\psi\rangle$ is SLOCC-maximal.*

SLOCC-equivalence is an equivalence relation, then equivalence class can be defined. This equivalence class is called SLOCC-equivalent class, or simply,

Name	Representative	Name	Representative
$\lvert\psi_0\rangle$	$\lvert 000\rangle$	$\lvert\psi_{12}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 112\rangle$
$\lvert\psi_1\rangle$	$\lvert 000\rangle + \lvert 011\rangle$	$\lvert\psi_{13}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 112\rangle + \lvert 120\rangle$
$\lvert\psi_2\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle$	$\lvert\psi_{14}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 120\rangle + \lvert 101\rangle$
$\lvert\psi_3\rangle$	$\lvert 000\rangle + \lvert 101\rangle$	$\lvert\psi_{15}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 120\rangle + \lvert 102\rangle$
$\lvert\psi_4\rangle$	$\lvert 000\rangle + \lvert 110\rangle$	$\lvert\psi_{16}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle$
$\lvert\psi_5\rangle$	$\lvert 000\rangle + \lvert 111\rangle$	$\lvert\psi_{17}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle$
$\lvert\psi_6\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 101\rangle$	$\lvert\psi_{18}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 112\rangle + \lvert 120\rangle$
$\lvert\psi_7\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 112\rangle$	$\lvert\psi_{19}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 120\rangle + \lvert 101\rangle$
$\lvert\psi_8\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 120\rangle$	$\lvert\psi_{20}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 122\rangle$
$\lvert\psi_9\rangle$	$\lvert 000\rangle + \lvert 101\rangle + \lvert 202\rangle$	$\lvert\psi_{21}\rangle$	$\lvert 000\rangle + \lvert 110\rangle + \lvert 220\rangle$
$\lvert\psi_{10}\rangle$	$\lvert 000\rangle + \lvert 111\rangle + \lvert 202\rangle$	$\lvert\psi_{22}\rangle$	$\lvert 000\rangle + \lvert 111\rangle + \lvert 220\rangle$
$\lvert\psi_{11}\rangle$	$\lvert 000\rangle + \lvert 111\rangle + \lvert 201\rangle$	$\lvert\mathcal{G}\rangle$	$\lvert 000\rangle + \lvert 111\rangle + \lvert 222\rangle$

Name	Representative
$\lvert\pi(\phi,\varphi,\chi,\psi)\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 1\phi\varphi\rangle + \lvert 2\chi\psi\rangle$
$\lvert\phi_0\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 202\rangle$
$\lvert\phi_1\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 110\rangle + \lvert 220\rangle$
$\lvert\varphi_1\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 212\rangle$
$\lvert\phi_2(\phi,\varphi)\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 2\phi\varphi\rangle$
$\lvert\varphi_2(\phi,\varphi)\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 112\rangle + \lvert 120\rangle + \lvert 2\phi\varphi\rangle$
$\lvert\phi_3(\phi,\varphi)\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 120\rangle + \lvert 101\rangle + \lvert 2\phi\varphi\rangle$
$\lvert\phi_4\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 202\rangle + \lvert 221\rangle$
$\lvert\psi_{23}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 210\rangle + \lvert 202\rangle$
$\lvert\phi_5\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 221\rangle + \lvert 210\rangle$
$\lvert s_0\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 112\rangle + \lvert 120\rangle + \lvert 202\rangle + \lvert 221\rangle$
$\lvert\phi_6\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 112\rangle + \lvert 120\rangle + \lvert 221\rangle + \lvert 210\rangle$
$\lvert\psi_{24}\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 120\rangle + \lvert 101\rangle + \lvert 221\rangle + \lvert 210\rangle$
$\lvert\phi_7\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 202\rangle + \lvert 221\rangle$
$\lvert\phi_8\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 210\rangle + \lvert 202\rangle$
$\lvert s_1\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 221\rangle + \lvert 210\rangle$
$\lvert w_0\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 202\rangle$
$\lvert\varphi_3\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 220\rangle$
$\lvert\phi_9\rangle$	$\lvert 000\rangle + \lvert 011\rangle + \lvert 022\rangle + \lvert 101\rangle + \lvert 112\rangle + \lvert 221\rangle$

Table 2.1: Representatives in SLOCC classes of tripartite qutrits

SLOCC class. When x is a representative of an SLOCC class, we use $\overline{x}$ to indicate the SLOCC class. It has been shown in [4] that N-partite systems $|\psi\rangle$ and $|\phi\rangle$ are SLOCC-equivalent iff there are regular matrices $L_1, \dots, L_n$ such that $|\psi\rangle = \bigotimes_{i=1}^{N} L_i|\phi\rangle$.

There are six SLOCC classes of tripartite qubits : $\overline{|000\rangle}$, $\overline{|000\rangle + |011\rangle}$, $\overline{|000\rangle + |101\rangle}$, $\overline{|000\rangle + |110\rangle}$, $\overline{|\mathrm{GHZ}\rangle}$, and $\overline{|\mathrm{W}\rangle}$. On the contrast, the SLOCC classes of tripartite qutrits in which we are interested are infinite. Using the inductive method introduced in [6], [9] has shown the SLOCC classes of tripartite qutrits. These classes are shown in Table 2.1. In the table, a column "Representative" is a representative of each class, and "Name" is a name we use to indicate a representative in this paper. ϕ, φ, χ, ψ is a unit vector of $\mathbb{C}^3$. Notice that $\overline{|\pi(\phi,\varphi,\chi,\psi)\rangle}$ indicates infinite number of SLOCC classes. $\overline{|\phi_{03}(\phi,\varphi)\rangle}$, $\overline{|\varphi_2(\phi,\varphi)\rangle}$ and $\overline{|\phi_{04}(\phi,\varphi)\rangle}$ also indicate infinite classes.

Chapter 3

Commutative Frobenius Algebra

In this chapter, firstly, we show the definitions of categories, monoidal categories and some related terms. Secondly, we define commutative Frobenius algebra on monoidal categories, and give graphical representation of this algebra. Thirdly, we give the special states which corresponds to the algebras, and finally, show classification of the algebras.

3.1 Category

Definition 3.1 (Category)**.** *Category C consists of*

1. *objects* $a, b, c, \ldots$
2. *arrows* $f, g, h, \ldots$
3. *four operations*
   (a) *domain : for each arrow f assign an object* $\mathrm{dom}\, f$
   (b) *codomain : for each arrow f assign an object* $\mathrm{cod}\, f$
   (c) *identity : for each object a assign an arrow* 1_a
   (d) *composition : for each pair of two arrows f and g such that* $\mathrm{cod}\, f = \mathrm{dom}\, g$ *assign an arrow $g \circ f$, which domain is* $\mathrm{dom}\, f$ *and codomain is* $\mathrm{cod}\, g$

An arrow f such that $a = \mathrm{dom} f$ and $b = \mathrm{cod} f$ is displayed as $a \xrightarrow{f} b$ or $f : a \to b$. An object a is displayed as $a \in C$ and an arrow f is displayed as $f \mathrm{in} C$. A category C satisfies three axioms:

1. *for any $f : a \to b$, $g : b \to c$, and $h : c \to d$, $(h \circ g) \circ f = h \circ (g \circ f)$*
2. *for any $f : a \to b$ and $g : b \to c$, $1_b \circ f = f$ and $g \circ 1_b = g$.*

We give examples of categories.

Example 3.2. Set *is a category which has all sets as objects and all functions between sets as arrows.*

Example 3.3. FdHilb *is a category which has all finite dimensional Hilbert space as objects and all linear functions between the spaces as arrows.*

In category theory, diagrams are often used. Edges of these diagrams are labelled by arrows and vertexes of them are labelled by objects.

Definition 3.4 (Commutative)**.** *Let v and v' be vertexes of a diagram. For any route from v to v' of the diagram, if compositions of labels of each line are same, diagram is commutative.*

Some arrows have the inverse arrows.

Definition 3.5 (Isomorphism)**.** *Let $f : a \to b$ be an arrow. If there is an arrow $g : b \to a$ such that $g \circ f = 1_a$ and $f \circ g = 1_b$, then f is invertible. g is written as f^{-1} and called the inverse of f. An invertible arrow is called an isomorphism.*

Example 3.6. *Isomorphisms of* **Set** *are bijective functions.*

When there are two categories, a new category called a product category can be made from them.

Definition 3.7 (Product Category)**.** *Let B and C are categories. A product category $B \times C$ is a category which objects are pairs (b, c) for any $b \in B$ and $c \in C$ and arrows are pairs $(f, g) : (b, c) \to (b', c')$ for any $f : b \to b'$ in B and $g : c \to c'$ in C. A composition of $(f', g') : (b', c') \to (b'', c'')$ and $(f, g) : (b, c) \to (b', c')$ is given as $(f' \circ f, g' \circ g) : (b, c) \to (b', c')$. An identity of (b, c) is $(1_b, 1_c)$.*

Some categories have some kind of relations. This relation can be considered as arrows between categories.

Definition 3.8 (Functor, Bifunctor)**.** *Let A, B, C, and D be categories. Functor $F : A \to B$ is a function which assigns any object $a \in A$ to $Fa \in B$ and any arrow g in A to Fg in B. This assignment reserves compositions and identities.*

Bifunctor $F : A \times C \to B$ is a functor from a product category.

Furthermore, "arrows" between functors can be defined.

Definition 3.9 (Natural transformation, Natural isomorphism)**.** *Let $F : A \to B$ and $G : A \to B$ be functors. Natural transformation $\gamma : F \dot{\to} G$ assign any object $a \in A$ to γ_a in $B : Fa \to Ga$ such that for any arrow $f : a \to b$ in B, the diagram*

$$\begin{tikzcd} Fa \arrow[r, "\gamma_a"] \arrow[d, "Ff"] & Ga \arrow[d, "Gf"] \\ Fb \arrow[r, "\gamma_b"] & Gb \end{tikzcd} \tag{3.1}$$

commute. Moreover, if for any object $a \in A$, γ_a is an isomorphism, then γ is called a natural isomorphism. A natural isomorphism is written as $\gamma : F \cong G$ or $\gamma_a : Fa \cong Ga$.

3.2 Symmetric Monoidal Category

To take state vectors, tensor product is needed. A category which has a tensor product is called a monoidal category. We use the symmetric version of the category.

Definition 3.10 (Symmetric Monoidal Category)**.** *Monoidal Category M consists of*

(i) a category C

(ii) a bifunctor named a tensor product $\otimes : C \times C \to C$

(iii) an object named a unit object $e \in C$

(iv) a natural isomorphism $\lambda_a : e \otimes a \cong a$
(v) a natural isomorphism $\rho_a : a \otimes e \cong a$
(vi) a natural isomorphism $\alpha_{a,b,c} : a \otimes (b \otimes c) \cong (a \otimes b) \otimes c$
M makes the following two diagrams commute :

$$\begin{tikzcd}
a \otimes ((b \otimes c) \otimes d \arrow[rr, "\alpha_{a,b\otimes c,d}"] & & (a \otimes (b \otimes c)) \otimes d \arrow[d, "\alpha_{a,b,c} \otimes 1_a"] \\
a \otimes (b \otimes (c \otimes d)) \arrow[u, "1_a \otimes \alpha_{b,c,d}"] \arrow[r, "\alpha_{a,b,c\otimes d}"] & (a \otimes b) \otimes (c \otimes d) \arrow[r, "\alpha_{a\otimes b,c,d}"] & ((a \otimes b) \otimes c) \otimes d
\end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd}
a \otimes (e \otimes b) \arrow[rr, "\alpha_{a,e,b}"] \arrow[dr, "\lambda_b"] & & (a \otimes e) \otimes b \arrow[dl, "\rho_a"] \\
& a \otimes b &
\end{tikzcd} \tag{3.3}$$

If M has a natural isomorphism $\gamma_{a,b} : a \otimes b \cong b \otimes a$ such that three diagrams

$$\begin{tikzcd}
a \otimes e \arrow[rr, "\gamma_{a,e}"] \arrow[dr, "\rho_a"] & & e \otimes a \arrow[dl, "\lambda_a"] \\
& a &
\end{tikzcd} \tag{3.4}$$

$$\begin{tikzcd}
a \otimes b \arrow[r, "\gamma_{a,b}"] \arrow[dr, "1_{a\otimes b}"] & b \otimes a \arrow[d, "\gamma_{b,a}"] \\
& a \otimes b
\end{tikzcd} \tag{3.5}$$

$$\begin{tikzcd}
(a \otimes b) \otimes c \arrow[r, "\gamma_{a\otimes b,c}"] \arrow[d, "\alpha^{-1}_{a,b,c}"] & c \otimes (a \otimes b) \arrow[d, "\alpha_{c,a,b}"] \\
a \otimes (b \otimes c) \arrow[d, "1_a \otimes \gamma_{b,c}"] & (c \otimes a) \otimes b \arrow[d, "\gamma_{c,a} \otimes 1_b"] \\
a \otimes (c \otimes b) \arrow[r, "\alpha_{a,c,b}"] & (a \otimes c) \otimes b
\end{tikzcd} \tag{3.6}$$

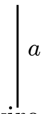
commute, then M is called a symmetric monoidal category.

Theorem 3.11 (Coherence Theorem [18]). *Any diagram which is composed by α, λ, ρ, γ, identities, and their tensor products commute.*

This theorem means that any object which is tensor product of $a_1, \ldots, a_n$, and e can be identifiable even if positions of bracket or e, or a number of e differ.

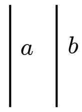
Example 3.12. FdHilb *can be a monoidal category where the tensor product is the usual tensor product and the unit object is $\mathbb{C}$.*

It is known that arrows on monoidal categories can be expressed graphically [13]. We assume implicitly the flow from top to bottom. An object a is written as a line.



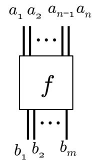
(3.7)

The unit object e is expressed as no-wire. Tensor product $a \otimes b$ is written as two lines.



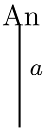
(3.8)

An arrow $f : a_1 \otimes \cdots \otimes a_n \to b_1 \otimes \cdots \otimes b_m$ is written as a box labelled f from input wires $a_1, \ldots, a_n$ to output wires $b_1, \ldots, b_m$.



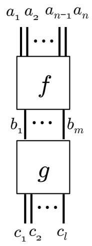
(3.9)

e is not written graphically and there exists the coherence theorem, so $\alpha_{a,b,c}$, λ_a, and ρ_a are not expressed graphically. An identity arrow 1_a is written as a wire a.



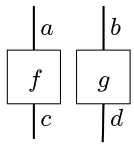
(3.10)

Composition $g \circ f$ of arrows $f : a_1 \otimes \cdots \otimes a_n \to b_1 \otimes \cdots \otimes b_m$ and $g : b_1 \otimes \cdots \otimes b_m \to c_1 \otimes \cdots \otimes c_l$ is expressed as vertical juxtaposition.



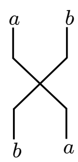
(3.11)

Two wires can be connected only if they are labelled by the same object. Tensor product $f \otimes g : a \otimes b \to c \otimes d$ of arrows $f : a \to c$ and $g : b \to d$ is expressed as horizontal juxtaposition.



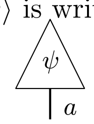
(3.12)

Furthermore, a natural isomorphism $\gamma_{a,b}$ of symmetric monoidal categories is expressed as an intersection.



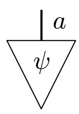
(3.13)

By these expressions, any arrows on monoidal categories can be represented graphically. Recalling that $|\psi\rangle$ can be regarded as a function from $\mathbb{C}$ to $\mathbb{C}^3$, $|\psi\rangle$ can be expressed graphically on monoidal categories **FdHilb**. Generally, a state vector $|\psi\rangle$ of a state space a can be regarded as a function from $\mathbb{C}$ to a. Especially, emphasising no-input, $|\psi\rangle$ is written as a triangle.



(3.14)

$\langle\psi|$ is written as a reversed triangle.



(3.15)

3.3 Commutative Frobenius Algebra

Using graphical representation on monoidal categories, any arrows on monoidal categories can be represented graphically. Therefore, if systems correspond to arrows on monoidal categories, then any system can be represented graphically. It has been shown in [17] that an algebra strictly correspond to some kind of systems.

Definition 3.13 (Commutative Frobenius Algebra)**.** *A Frobenius algebra F on a monoidal category M consists of*

(i) an object $a \in M$
(ii) multiplication $\mu : a \otimes a \to a$
(iii) a unit $\eta : e \to a$
(iv) comultiplication $\delta : a \to a \otimes a$
(v) a counit : $\epsilon : a \to e$
where e is the unit object of M. F makes the following diagrams commute:

$$\begin{tikzcd}
& a \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\
a \otimes a \arrow[d, "1_a \otimes \delta"] & & a \otimes a \arrow[d, "\delta \otimes 1_a"] \\
a \otimes (a \otimes a) \arrow[rr, "\alpha_{a,a,a}"] & & (a \otimes a) \otimes a
\end{tikzcd} \tag{3.16}$$

$$\begin{tikzcd}
a \otimes a \arrow[d, "\epsilon \otimes 1_a"] & a \arrow[l, "\delta"'] \arrow[r, "\delta"] \arrow[d, "1_a"] & a \otimes a \arrow[d, "1_a \otimes \epsilon"] \\
e \otimes a \arrow[r, "\lambda_a"] & a & a \otimes e \arrow[l, "\rho_a"']
\end{tikzcd} \tag{3.17}$$

$$\begin{tikzcd}
a \otimes (a \otimes a) \arrow[rr, "\alpha_{a,a,a}"] \arrow[d, "1_a \otimes \mu"] & & (a \otimes a) \otimes a \arrow[d, "\mu \otimes 1_a"] \\
a \otimes a \arrow[dr, "\mu"] & & a \otimes a \arrow[dl, "\mu"] \\
& a &
\end{tikzcd} \tag{3.18}$$

$$\begin{tikzcd}
e \otimes a \arrow[d, "\eta \otimes 1_a"] & a \arrow[l, "\lambda_a^{-1}"'] \arrow[r, "\rho_a^{-1}"] \arrow[d, "1_a"] & a \otimes e \arrow[d, "1_a \otimes \eta"] \\
a \otimes a \arrow[r, "\mu"] & a & a \otimes a \arrow[l, "\mu"']
\end{tikzcd} \tag{3.19}$$

$$
\begin{tikzcd}
(a \otimes a) \otimes a \arrow[rr, "\alpha^{-1}_{a,a,a}"] & & a \otimes (a \otimes a) \arrow[d, "1_a \otimes \mu"] \\
a \otimes a \arrow[u, "\delta \otimes 1_a"] \arrow[r, "\mu"] \arrow[d, "1_a \otimes \delta"'] & a \arrow[r, "\delta"] & a \otimes a \\
a \otimes (a \otimes a) \arrow[rr, "\alpha_{a,a,a}"] & & (a \otimes a) \otimes a \arrow[u, "\mu \otimes 1_a"']
\end{tikzcd}
\tag{3.20}
$$

If F makes the following diagrams commute :

$$
\begin{tikzcd}
a \otimes a \arrow[r, "\gamma_{a,a}"] \arrow[dr, "\mu"] & a \otimes a \arrow[d, "\mu"] \\
& a
\end{tikzcd}
\tag{3.21}
$$

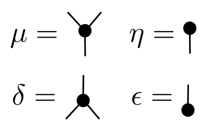
$$
\begin{tikzcd}
a \arrow[d, "\delta"] \arrow[dr, "\delta"] & \\
a \otimes a \arrow[r, "\gamma_{a,a}"] & a \otimes a
\end{tikzcd}
\tag{3.22}
$$

then F is called a commutative Frobenius algebra (CFA).

We can get a CFA $(\mathcal{H}, \mu, \delta, \eta, \epsilon)$ on a monoidal category **FdHilb**.

Definition 3.14 (F-graph)**.** *A F-graph of a CFA F is an arrow which is composed by μ, δ, ϵ, δ, $\alpha_{a,a,a}$, ρ_a, λ_a, $\gamma_{a,a}$, 1_a, and their tensor products.*

Domain-codomain pairs of all the components of a CFA differ from each other. Therefore, without label of each arrow, we can write a CFA graphically, as follows.



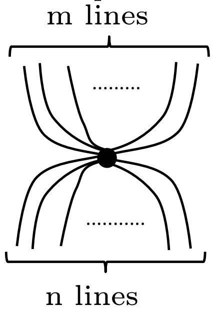
(3.23)

Using the representation of CFAs and monoidal categories, any F-graph can be represented graphically. This representation is determined by their topological properties.

Theorem 3.15 ([17])**.** *For any F-graphs f and g, graphical representations of which are connected, if these representation have the same numbers of inputs, outputs, and loops, then they are same. The number of loops means the maximum number of wire which can be removed without destroying connection of representation.*

We give notations of some F-graphs for simplification. A notation, called spider notation in [17] is used for F-graphs which do not have any loop. A

F-graph which has m inputs and n outputs is written as the following.



(3.24)

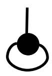
F-graph which has no input and which has no output is written as the following.

(3.25)

Moreover, some F-graphs which have exactly one loop have special notations. A F-graph which has a loop, no input, and an output is written as the following.

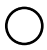
(3.26)

Similarly, a F-graph which has a loop, an input, and no output is expressed as the following.



(3.27)

A F-graph which has a loop, no input, and no output is expressed as a circle.

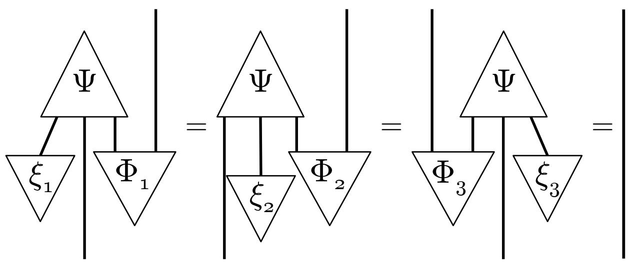


(3.28)

3.4 Frobenius State

CFAs correspond to some kind of states, called Frobenius states. These states require some properties called strong SLOCC-maximality and symmetry.

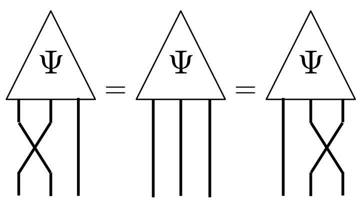
Definition 3.16 (Strong SLOCC-maximality). *Let $|\Psi\rangle$ be a tripartite state. If there are $\langle\Phi_i|$ and $\langle\xi_i|$ such that*



(3.29)

then $|\Psi\rangle$ is strongly SLOCC-maximal.

Definition 3.17 (Symmetric State). *Let $|\Psi\rangle$ be a tripartite state. If $|\Psi\rangle$ satisfies the following equations :*



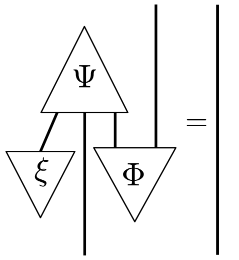
(3.30)

then $|\Psi\rangle$ is a symmetric state.

Similarly, N-partite symmetric states are defined.

For symmetric states, the definition of strong SLOCC-maximality can be rewritten simply as the following proposition.

Proposition 3.18 ([17]). *Let $|\Psi\rangle$ be a tripartite symmetric state. $|\Psi\rangle$ is strongly SLOCC-maximal, if there are $\langle\Phi|$ and $\langle\xi|$ such that*

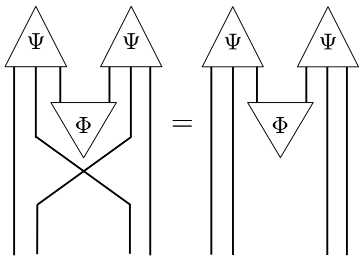


(3.31)

Obviously, for any strongly SLOCC-maximal and symmetric state $|\Psi\rangle$, if $\langle\xi|$ is given, then $\langle\Phi|$ is uniquely determined. Conversely, if $\langle\Phi|$ is given, then $\langle\xi|$ is uniquely determined. By the uniqueness, we write a Frobenius state $|\Psi\rangle$ with $\langle\xi|$ to indicate the trio $|\Psi\rangle$, $\langle\Phi|$, and $\langle\xi|$ such that they satisfy the Frobenius conditions. Notice that for a Frobenius state $|\Psi\rangle$, there is generally more than one pair ($\langle\xi|$, $\langle\Phi|$) such that $|\Psi\rangle$, $\langle\Phi|$, and $\langle\xi|$ satisfy the Frobenius conditions.

Frobenius states also require not only symmetry but also strong symmetry to correspond to CFAs.

Definition 3.19 (Strong Symmetry). *Let $|\Psi\rangle$ be a tripartite state. If $|\Psi\rangle$ is a symmetric state and there is $\langle\Phi|$ such that*



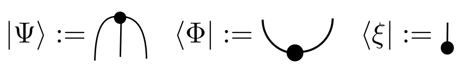
(3.32)

then $|\Psi\rangle$ is strongly symmetric.

Definition 3.20 (Frobenius State). *Let $|\Psi\rangle$ be a tripartite state. If there are $\langle\Phi|$ and $\langle\xi|$ and they satisfies (3.31) and (3.32), then $|\Psi\rangle$ is a Frobenius state.*

Notice that this definition requires the satisfaction of the equations with same $\langle\Phi|$ and $\langle\xi|$. Frobenius state and CFAs are connected strictly. The connection is given by the following theorems.

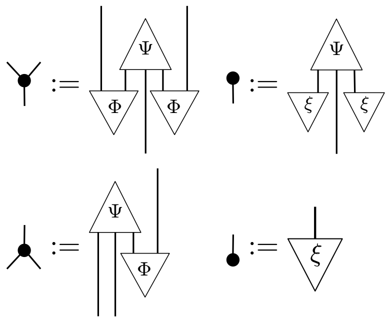
Theorem 3.21 ([17]). *For any CFA,*



(3.33)

$|\Psi\rangle$ is a Frobenius state with $\langle\Phi|$ and $|\xi\rangle$.

Theorem 3.22 ([17])**.** *Any Frobenius state defines a CFA* ($\mathcal{H}$, , , ,) *as*



$$(3.34)$$

Theorem 3.21 shows how to make a Frobenius state from a CFA, and Theorem 3.22 shows the inverse. Notice that a CFA induced by a Frobenius State with $\langle\xi|$ and $\langle\Phi|$ which a CFA induces is same to the original CFA. That means that any CFA correspond to a Frobenius state strictly. To give examples, we show CFAs which correspond to tripartite qubits. These CFAs defined on **FdHilb**. In tripartite qubits, there are six SLOCC classes : $\overline{|000\rangle}$, $\overline{|000\rangle+|011\rangle}$, $\overline{|000\rangle+|101\rangle}$, $\overline{|000\rangle+|110\rangle}$, $\overline{|\mathrm{GHZ}\rangle}$, $\overline{|\mathrm{W}\rangle}$. Obviously the first four classes are not strongly SLOCC-maximal. On the contrast, $|\mathrm{GHZ}\rangle$ and $|\mathrm{W}\rangle$, which are defined in (2.8) and (2.9) are Frobenius states, so these states correspond CFAs.

Example 3.23. $|GHZ\rangle$ *with* $\langle\xi| := \langle 0| + \langle 1|$ *induces a CFA*

$$\begin{aligned} &:= |0\rangle\langle 00| + |1\rangle\langle 11| \quad := |0\rangle + |1\rangle \\ &:= |00\rangle\langle 0| + |11\rangle\langle 1| \quad := \langle 0| + \langle 1| \end{aligned} \tag{3.35}$$

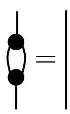
Example 3.24. $|W\rangle$ *with* $\langle\xi| := \langle 0|$ *induces a CFA*

$$\begin{aligned} &:= |0\rangle\langle 01| + |0\rangle\langle 10| + |1\rangle\langle 11| \quad := |1\rangle \\ &:= |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 1| \quad := \langle 0| \end{aligned} \tag{3.36}$$

3.5 Classification of Tripartite Qubits

To classify tripartite qubit, two kinds of CFAs are defined.

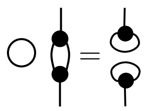
Definition 3.25 (Special Commutative Frobenius Algebra)**.** *A commutative Frobenius algebra which satisfies the following equation :*



$$(3.37)$$

is a special commutative Frobenius algebra (SCFA).

Definition 3.26 (Anti-special Commutative Frobenius Algebra)**.** *A commutative Frobenius algebra which satisfies the following equation :*



$$(3.38)$$

is an anti-special commutative Frobenius algebra (ACFA).

These algebras are topologically different from each other. Using simple calculation, it is obvious that a CFA (3.35) is an SCFA, and a CFA (3.36) is an ACFA. For distinction, an SCFA is expressed by a white dot and an ACFA is expressed by a black dot . It has been shown in [17] that these two kinds of CFAs strictly correspond to two SLOCC classes of tripartite qubits.

Theorem 3.27 ([17])**.** *Let $|\Psi\rangle$ be a Frobenius state. $|\Psi\rangle$ is SLOCC-equivalent to GHZ state if and only if there is $\langle\xi|$ such that $|\Psi\rangle$ with $\langle\xi|$ induces an SCFA. $|\Psi\rangle$ is SLOCC-equivalent to W state if and only if there is $\langle\xi|$ such that $|\Psi\rangle$ with $\langle\xi|$ induces an ACFA.*

Chapter 4

Qutrits and Commutative Frobenius Algebras

In this chapter, firstly, we enumerate SLOCC classes which do not correspond to any CFA. Secondly, we define three CFAs which correspond to Frobenius states, classify the CFAs by graphical equations, and prove that the classification corresponds to the three CFAs strictly. Finally, we show how compose any qutrit graphically.

4.1 Non-Maximal Class

Frobenius states require strong SLOCC-maximality and strong symmetry. First, we use this first condition to judge which classes do not have SLOCC-maximal states. Strong SLOCC-maximality requires a tripartite qutrit to be full rank in every single qutrits. Some SLOCC classes do not include strongly SLOCC-maximal states. Absence of the states in an SLOCC class decides the property of the class. We call an SLOCC class which do not include any strongly SLOCC-maximal state *non-maximal class.*

Lemma 4.1. *For any tripartite SLOCC class X, if X include a qutrit which does not strongly SLOCC-maximal, then X is a non-maximal class.*

Proof. Let $|\phi\rangle$ be a qutrit which are SLOCC-equivalent to a strongly SLOCC-maximal qutrit $|\psi\rangle$. $|\psi\rangle$ has $\langle\xi_i|$ and $\langle\Phi_i|$ $(i \in \{1,2,3\})$ such that they satisfies the SLOCC-maximal conditions. Because $|\phi\rangle$ and $|\psi\rangle$ are SLOCC-equivalent, there are matrices L_1, L_2, and L_3 such that $|\phi\rangle = (L_1 \otimes L_2 \otimes L_3)|\psi\rangle$. $\langle\xi'_i| := \langle\xi_i| \circ L_i^{-1}$ and $\langle\Phi'_i| := \langle\Phi_i| \circ (L_j^{-1} \otimes L_k^{-1})$ with $i, j, k \in \{1,2,3\}$ which differ from each other. $|\phi\rangle$, $\langle\xi'_i|$, and $\langle\Phi'_i|$ satisfies the SLOCC-maximal conditions. □

Frobenius states require strong SLOCC-maximality, so a non-maximal class does not have Frobenius states. By simple calculation, it is proved that for any $i \in \{0,\ldots,24\}$, $\overline{|\psi_i\rangle}$ is a non-maximal classes.

In $\overline{|\pi(\phi,\varphi,\chi,\psi)\rangle}$, if $|\phi\rangle$ and $|\chi\rangle$ can be expressed as $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\chi\rangle = \gamma|0\rangle + \delta|1\rangle$ by some complex numbers α, β,γ, and δ, then this class does not include strongly SLOCC-maximal state. The same can be said for $|\varphi\rangle$ and $|\psi\rangle$.

4.2 Non-Symmetric Class

Next, we use the second condition, strong symmetry. Many SLOCC classes include strongly SLOCC-maximal states but a few classes include symmetric states.

We will call an SLOCC class in which any symmetric qutrit does not belong *non-symmetric class.* We will use a lemma to show which classes are non-symmetric classes.

Lemma 4.2. *For any permutation P, if a qutrit $|\phi\rangle$ is SLOCC-equivalent to a symmetric qutrit $|\psi\rangle$, $P|\phi\rangle$ is SLOCC-equivalent to $|\phi\rangle$.*

Applying this lemma to representatives, we can prove that for any $i \in \{0, \ldots, 9\}$, $\overline{|\phi_i\rangle}$ is a non-symmetric class.

In addition to Lemma 4.2, any permutation can be represented in a 3×3 matrix.

Lemma 4.3. *For any N-partite qutrit $|\phi\rangle$ and any permutation P between an ith qutrit and a jth qutrit, if $|\phi\rangle$ is SLOCC-equivalent to an N-partite symmetric qutrit $|\psi\rangle$, then there is a 3×3 regular matrix L and*

$$M_k = \begin{cases} L & (k = i) \\ L^{-1} & (k = j) \\ I & (otherwise) \end{cases} \tag{4.1}$$

such that $P|\phi\rangle = (\otimes_{k=1}^{N} M_k)|\phi\rangle$. I is an identity matrix.

Proof. Let $|\phi\rangle$ be a qutrit which SLOCC-equivalent to a symmetric qutrit $|\psi\rangle$. There are regular matrices L_k such that $|\psi\rangle = \otimes_{k=1}^{N} L_k|\phi\rangle$. Let F_k be functions such that

$$F_k := \begin{cases} L_j^{-1} & (k = i) \\ L_i^{-1} & (i = j) \\ L_k^{-1} & (otherwise) \end{cases} \tag{4.2}$$

It satisfies $P|\phi\rangle = \otimes_{k=1}^{N} F_k|\psi\rangle$. Let $L = L_j^{-1} \circ L_i$, then M_k which are defined in this lemma satisfy $P|\phi\rangle = \otimes_{k=1}^{N}(F_k \circ L_k)|\phi\rangle = \otimes_{k=1}^{N} M_k|\phi\rangle$. □

Using these lemmas, we can consider all classes expressed as $\overline{|\pi(\phi, \varphi, \chi, \psi)\rangle}$. We pick up an SLOCC-maximal class $\overline{|\pi(\phi, \varphi, \chi, \psi)\rangle}$. By the above argument, we assume that $|\phi\rangle$ or $|\chi\rangle$ cannot be expressed as $\alpha|0\rangle + \beta|1\rangle$. When $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$ by non-zero number γ, we can get a state $|\pi(\phi', \varphi, |2\rangle, \psi)\rangle$ in this class by the regular matrix which convert $|\chi\rangle$ to $|2\rangle$, $|0\rangle$ and $|1\rangle$ to themselves. When $|\chi\rangle = \alpha|0\rangle + \beta|1\rangle$, we can get a state $|\pi(\chi', \psi, |2\rangle, \varphi)\rangle$. By a conversion like above, it is confirmed that $|\pi(\phi', \varphi', |2\rangle, |2\rangle)\rangle$ or $|\pi(\phi', |2\rangle, |2\rangle, \alpha|0\rangle + \beta|1\rangle)\rangle$ belongs to $\overline{|\pi(\phi, \varphi, \chi, \psi)\rangle}$.

We examine these two cases. Firstly, we assume that $|\pi(\phi', |2\rangle, |2\rangle, \psi')\rangle$ belongs to the class such that $|\psi'\rangle = \alpha|0\rangle + \beta|1\rangle$. Consider a permutation between the second and third qutrits. By Lemma 4.3, there is a regular matrix L such that $|\pi(\phi', |2\rangle, |2\rangle, \psi')\rangle = (I \otimes L \otimes L^{-1})|\pi(\phi', |2\rangle, |2\rangle, \psi')\rangle$. This matrix L satisfies two equations below:

$$\begin{aligned} (L \otimes L^{-1})|00\rangle + |11\rangle &= |00\rangle + |11\rangle & (4.3) \\ (L \otimes L^{-1})(\alpha|20\rangle + \beta|21\rangle) &= \alpha|02\rangle + \beta|12\rangle & (4.4) \end{aligned}$$

L and L^{-1} can be written as

$$L = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} \tag{4.5}$$

$$L^{-1} = \begin{pmatrix} p & q & r \\ s & t & u \\ v & w & x \end{pmatrix} \tag{4.6}$$

Substituting (4.5) and (4.6) for (4.3), $\alpha = c(g\alpha + h\beta)$ and $\beta = f(g\alpha + h\beta)$. However, substituting (4.5) and (4.6) for (4.4), $g = h = 0$, because L is a regular matrix. Therefore, $\alpha = \beta = 0$. As a result, $\overline{|\pi(\phi', |2\rangle, |2\rangle, \psi')\rangle}$ is a non-symmetric class.

Secondly, we consider $|\pi(\phi', \varphi', |2\rangle, |2\rangle)\rangle$. $|\phi'\rangle$ and $|\varphi'\rangle$ are

$$\begin{aligned} |\phi'\rangle &= \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle & (4.7) \\ |\varphi'\rangle &= \delta|0\rangle + \eta|1\rangle + \theta|2\rangle & (4.8) \end{aligned}$$

Let L be a regular matrix such that $|\pi(\phi', \varphi', |2\rangle, |2\rangle)\rangle$ is equal to $(I \otimes L \otimes L^{-1})|\pi(\phi', \varphi', |2\rangle, |2\rangle)\rangle$. L and L^{-1} are expressed as (4.5) and (4.6) respectively. This matrix L satisfies two equations below:

$$\begin{aligned} (L \otimes L^{-1})|00\rangle + |11\rangle &= |00\rangle + |11\rangle & (4.9) \\ (L \otimes L^{-1})|\phi'\varphi'\rangle &= |\varphi'\phi'\rangle & (4.10) \\ (L \otimes L^{-1})|22\rangle &= |22\rangle & (4.11) \end{aligned}$$

By the above arguments and the same arguments about (4.9), it is derived that $g = h = v = w = 0$. Then, considered (4.10), $\gamma = 0$ if and only if $\theta = 0$. Calculating L of the permutation between the 1st and 2nd qutrits, it is indicated that $\gamma = \theta = 0$.

We can check in the same way that other classes $\overline{|\varphi_1\rangle}$, $\overline{|\varphi_2\rangle}$, and $\overline{|\varphi_3\rangle}$ are non-symmetric classes.

Now, we know all of non-symmetric classes. The rest classes are $\overline{|\mathcal{G}\rangle}$, $\overline{|w_0\rangle}$, $\overline{|s_0\rangle}$, $\overline{|s_1\rangle}$, $\overline{|\pi(\phi', \varphi', |2\rangle, |2\rangle)\rangle}$. Respectively, there are symmetric states in the classes : $|\mathcal{G}\rangle$, $|\mathcal{W}\rangle := |002\rangle + |011\rangle + |020\rangle + |101\rangle + |110\rangle + |200\rangle$, $|s_2\rangle := |000\rangle + |012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle$, $|s_3\rangle := |012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle$, $|\mathcal{I}\rangle := |001\rangle + |010\rangle + |100\rangle + |222\rangle$.

For $|\pi(\phi', \varphi', |2\rangle, |2\rangle)\rangle$, there are two cases which are SLOCC-equivalent to (i) $|000\rangle + |011\rangle + |100\rangle + |222\rangle$, (ii) $|000\rangle + |011\rangle + |101\rangle + |222\rangle$. First case is the same class to $|\mathcal{G}\rangle$. In the second class, there is a symmetric state $|\mathcal{I}\rangle$.

4.3 Frobenius Class

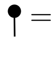
We will call an SLOCC class which includes a Frobenius state a *Frobenius class.* Now, we know only five classes include symmetric states. In order to restrict the classes to Frobenius classes, we assume that a proposition is true.

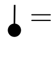
Conjecture. *For any symmetric tripartite qutrits $|\phi\rangle$ and $|\psi\rangle$, if they are SLOCC-equivalent, then there is a 3×3 matrix L such that $|\phi\rangle = (L \otimes L \otimes L)|\psi\rangle$.*

When two symmetric states belong to the same SLOCC class, they connected a function which can be written as tensor product of three regular matrices. It is natural to consider that this function is also symmetric. In the case of qubit, this conjecture is proved in [8].

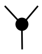
Because of the conjecture, any symmetric state SLOCC-equivalent to a Frobenius state is also a Frobenius state. Then, we check only a symmetric qutrit in all symmetric classes to judge whether the classes are Frobenius classes or not. By simple calculation, we can get three Frobenius states :

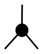
$|\mathcal{G}\rangle$ with $\langle\xi| := \langle 0| + \langle 1| + \langle 2|$:

 $= |0\rangle\langle 00| + |1\rangle\langle 11| + |2\rangle\langle 22|$  $= |0\rangle + |1\rangle + |2\rangle$

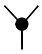
 $= |00\rangle\langle 0| + |11\rangle\langle 1| + |22\rangle\langle 2|$  $= \langle 0| + \langle 1| + \langle 2|$ (4.12)

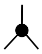
$|\mathcal{W}\rangle$ with $\langle\xi| := \langle 0|$:

 $= |0\rangle\langle 02| + |0\rangle\langle 11| + |0\rangle\langle 20| + |1\rangle\langle 12| + |1\rangle\langle 21| + |2\rangle\langle 22|$  $= |2\rangle$

 $= |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 1| + |02\rangle\langle 2| + |11\rangle\langle 2| + |20\rangle\langle 2|$  $= \langle 0|$ (4.13)

$|\mathcal{I}\rangle$ with $\langle\xi| := \langle 0| + \langle 2|$:

 $= |0\rangle\langle 01| + |0\rangle\langle 10| + |1\rangle\langle 11| + |2\rangle\langle 22|$  $= |1\rangle + |2\rangle$

 $= |00\rangle\langle 0| + |01\rangle\langle 1| + |10\rangle\langle 1| + |22\rangle\langle 2|$  $= \langle 0| + \langle 2|$ (4.14)

We call these algebras $\mathcal{G}$, $\mathcal{W}$, and $\mathcal{I}$ respectively.

However, the other two classes are not Frobenius classes. Firstly, we consider $|s_2\rangle$. Let $\langle\xi| = \alpha\langle 0| + \beta\langle 1| + \gamma\langle 2|$, then $\langle\Phi| = \frac{1}{\alpha(\alpha^2-2\beta\gamma)}(\alpha^2\langle 00| - \alpha\beta\langle 10| - \alpha\gamma\langle 20| - \alpha\beta\langle 01| + \beta^2\langle 11| + (\alpha^2-\beta\gamma)\langle 21| - \alpha\gamma\langle 02| + (\alpha^2-\beta\gamma)\langle 12| + \gamma^2\langle 22|)$. Calculating (3.32), we know $|s_2\rangle$ do not have strong symmetry.

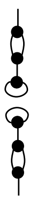
Secondly, consider $|s_3\rangle$. Similarly, we let $\langle\xi| = \alpha\langle 0| + \beta\langle 1| + \gamma\langle 2|$. $\langle\Phi|$ which satisfies (3.31) is $-\frac{\alpha}{2\beta\gamma}\langle 00| + \frac{1}{2\gamma}\langle 01| + \frac{1}{2\beta}\langle 02| + \frac{1}{2\gamma}\langle 10| - \frac{\beta}{2\alpha\gamma}\langle 11| + \frac{1}{2\alpha}\langle 12| + \frac{1}{2\beta}\langle 20| + \frac{1}{2\alpha}\langle 21| - \frac{\gamma}{2\alpha\beta}\langle 22|$. These $\langle\xi|$, $\langle\Phi|$, and $|s_3\rangle$ do not have strong symmetry.

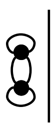
As a result, we got three Frobenius classes, and proved other classes are not Frobenius classes.

4.4 Classification of Commutative Frobenius Algebras

We can judge which class is an SCFA or an ACFA by calculation of . We can check smoothly $\mathcal{G}$ is an SCFA and $\mathcal{W}$ is an ACFA, but $\mathcal{I}$ is neither an SCFA nor an ACFA. Then, we define *intermediate special commutative Frobenius algebra.*

Definition 4.4 (ISCFA). *A commutative Frobenius algebra which satisfies the following two equations :*

 $=$  (4.15)

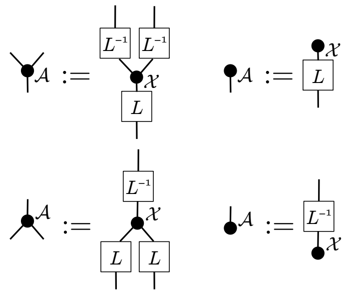
 $=$ (4.16)

is an intermediate special commutative Frobenius algebra (ISCFA).

An ISCFA is expressed by a white dot with a central small black dot . We can immediately verify that $\mathcal{I}$ is an ISCFA, but $\mathcal{G}$ and $\mathcal{W}$ are not so. Moreover, we can prove these three kinds of algebras exactly correspond to three Frobenius classes.

Theorem 4.5. *If a Frobenius state $|\psi\rangle$ is SLOCC-equivalent to a Frobenius state $|\phi\rangle$ which induces an SCFA, an ACFA, and an ISCFA with $\langle\xi|$, then there is $\langle\xi'|$ such that $|\psi\rangle$ with $\langle\xi'|$ induces an SCFA, an ACFA, and an ISCFA respectively.*

Proof. Let $\mathcal{X} = (\mathbb{C}^3, {}_{\mathcal{X}}, {}^{\mathcal{X}}, {}_{\mathcal{X}}, {}_{\mathcal{X}})$ be an SCFA induced by $|\phi\rangle$ with $\langle\xi|$. There is a regular matrix L such that $|\psi\rangle = (L \otimes L \otimes L)|\phi\rangle$. $|\psi\rangle$ with $\langle\xi'| := \langle\xi| \circ L^{-1}$ induces a CFA $\mathcal{A} = (\mathbb{C}^3, {}_{\mathcal{A}}, {}^{\mathcal{A}}, {}_{\mathcal{A}}, {}^{\mathcal{A}})$:



(4.17)

$\mathcal{A}$ satisfies the SCFA condition.

Other cases are proved by the same way. □

Corollary 4.6. *If a Frobenius state $|\psi\rangle$ is SLOCC-equivalent to $|\mathcal{G}\rangle$, $|\mathcal{W}\rangle$, and $|\mathcal{I}\rangle$, then there is $\langle\xi'|$ such that $|\psi\rangle$ with $\langle\xi'|$ induces an SCFA, an ACFA, and an ISCFA respectively.*

Theorem 4.7. *Let $|\psi\rangle$ be a Frobenius state. If there is $\langle\xi|$ such that $|\psi\rangle$ with $\langle\xi|$ induces an SCFA, then $|\psi\rangle$ is SLOCC-equivalent to $|\mathcal{G}\rangle$.*

Proof. This theorem can be proved by the same way in [17]. It is proved in [19] that three copyable vectors of δ form an orthogonal basis for $\mathbb{C}^3$. Then, we can get a regular matrix L which convert $|0\rangle$, $|1\rangle$, and $|2\rangle$ to the copyable vectors respectively. This L satisfies

$$|\psi\rangle = (L \otimes L \otimes L)|\mathcal{G}\rangle \tag{4.18}$$

□

Theorem 4.8. *Let $|\psi\rangle$ be a Frobenius state. If there is $\langle\xi|$ such that $|\psi\rangle$ with $\langle\xi|$ induces an ACFA, then $|\psi\rangle$ is SLOCC-equivalent to $|\mathcal{W}\rangle$.*

Proof. $|\psi\rangle$ is a Frobenius state, so $|\psi\rangle$ is SLOCC-equivalent to one of the states $|\mathcal{G}\rangle$, $|\mathcal{W}\rangle$, and $|\mathcal{I}\rangle$.

First, assume $|\psi\rangle$ be SLOCC-equivalent to $|\mathcal{G}\rangle$. Because of Theorem 4.5, there is $\langle\xi'|$ such that $|\mathcal{G}\rangle$ with $\langle\xi'|$ induces an ACFA. Let $\langle\xi'| := \alpha\langle 0| + \beta\langle 1| + \gamma\langle 2|$ with arbitrary complex number α, β, and γ. By the strongly SLOCC-maximal condition (3.31), α, β, and γ are restricted to nonzero. $|\mathcal{G}\rangle$ with $\langle\xi'|$ induces a CFA

$$\begin{aligned} &= \tfrac{1}{\alpha^2}|0\rangle\langle 00| + \tfrac{1}{\beta^2}|1\rangle\langle 11| + \tfrac{1}{\gamma^2}|2\rangle\langle 22| \qquad && = \alpha^2|0\rangle + \beta^2|1\rangle + \gamma^2|2\rangle \\ &= \tfrac{1}{\alpha}|00\rangle\langle 0| + \tfrac{1}{\beta}|11\rangle\langle 1| + \tfrac{1}{\gamma}|22\rangle\langle 2| && = \alpha\langle 0| + \beta\langle 1| + \gamma\langle 2|. \end{aligned} \tag{4.19}$$

This CFA does not satisfy the ACFA condition. This contradicts to the assumption.

Next, assume $|\psi\rangle$ be SLOCC-equivalent to $|\mathcal{I}\rangle$. $|\mathcal{I}\rangle$ with $\langle\xi'|$ induces a CFA

$$\begin{aligned} &= \tfrac{1}{\alpha^2}|0\rangle\langle 01| + \tfrac{1}{\alpha^2}|0\rangle\langle 10| + \tfrac{1}{\alpha^2}|1\rangle\langle 11| - \tfrac{2\beta}{\alpha^3}|0\rangle\langle 11| + \tfrac{1}{\gamma^2}|2\rangle\langle 22| \\ &= \tfrac{1}{\alpha}|00\rangle\langle 0| + \tfrac{1}{\alpha}|01\rangle\langle 1| + \tfrac{1}{\alpha}|10\rangle\langle 1| - \tfrac{\beta}{\alpha^2}|00\rangle\langle 1| + \tfrac{1}{\gamma}|22\rangle\langle 2| \\ &= 2\alpha\beta|0\rangle + \alpha^2|1\rangle + \gamma^2|2\rangle \\ &= \alpha\langle 0| + \beta\langle 1| + \gamma\langle 2|. \end{aligned} \tag{4.20}$$

with complex numbers α, β, and γ such that α and γ are nonzero. The CFA is not an ACFA. It is contradiction.

Therefore, $|\psi\rangle$ is SLOCC-equivalent to $|\mathcal{W}\rangle$. $\square$

Theorem 4.9. *Let $|\psi\rangle$ be a Frobenius state. If there is $\langle\xi|$ such that $|\psi\rangle$ with $\langle\xi|$ induces an ISCFA, then $|\psi\rangle$ is SLOCC-equivalent to $|\mathcal{I}\rangle$.*

Proof. It can be proved by the same way of the above theorem. First, assume $|\psi\rangle$ be SLOCC-equivalent to $|\mathcal{G}\rangle$. (4.19) induced by $|\mathcal{G}\rangle$ with $\langle\xi'|$ does not satisfy (4.15).

Second, assume $|\psi\rangle$ be SLOCC-equivalent to $|\mathcal{W}\rangle$. $|\mathcal{W}\rangle$ with $|\xi\rangle$ induces


$$= \tfrac{1}{\alpha^2}|0\rangle\langle 02| + \tfrac{1}{\alpha^2}|0\rangle\langle 11| - \tfrac{2\beta}{\alpha^3}|0\rangle\langle 12| + \tfrac{1}{\alpha^2}|1\rangle\langle 12| + \tfrac{1}{\alpha^2}|0\rangle\langle 20| - \tfrac{2\beta}{\alpha^3}|0\rangle\langle 21| + \tfrac{1}{\alpha^2}|1\rangle\langle 21| + \tfrac{3\beta^2-2\alpha\gamma}{\alpha^4}|0\rangle\langle 22| - \tfrac{2\beta}{\alpha^3}|1\rangle\langle 22| + \tfrac{1}{\alpha^2}|2\rangle\langle 22|$$

$$= \tfrac{1}{\alpha}|00\rangle\langle 0| - \tfrac{\beta}{\alpha^2}|00\rangle\langle 1| + \tfrac{1}{\alpha}|01\rangle\langle 1| + \tfrac{1}{\alpha}|10\rangle\langle 1| - \tfrac{\alpha\gamma-\beta^2}{\alpha^3}|00\rangle\langle 2| - \tfrac{\beta}{\alpha^2}|01\rangle\langle 2| - \tfrac{\beta}{\alpha^2}|10\rangle\langle 2| + \tfrac{1}{\alpha}|02\rangle\langle 2| + \tfrac{1}{\alpha}|11\rangle\langle 2| + \tfrac{1}{\alpha}|20\rangle\langle 2| \tag{4.21}$$

$$= (2\alpha\gamma+\beta^2)|0\rangle + 2\alpha\beta|1\rangle + \alpha^2|2\rangle$$

$$= \alpha\langle 0| + \beta\langle 1| + \gamma\langle 2|.$$

with arbitrary complex numbers β and γ, and a nonzero complex number α. The CFA does not satisfy (4.16).

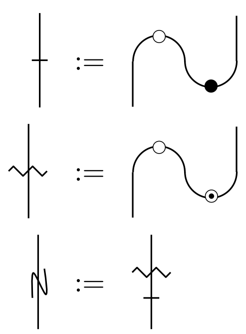
Therefore, $|\psi\rangle$ is SLOCC-equivalent to $|\mathcal{I}\rangle$. $\square$

4.5 Multiple Commutative Frobenius Algebras

It is shown in [17] that some pair of a two dimensional SCFA and ACFA which they called GHZ/W-pair is the ability to represent N-partite qubits. In this section, we show an SCFA, an ACFA, and an ISCFA have the ability to represent any qutrit.

We define a CFA trio $\mathcal{T} := (\mathcal{G}, \mathcal{W}, \mathcal{I})$, and the following arrows.

Definition 4.10 (Tick, Knurl, Wave)**.**



(4.22)

(4.23)

(4.24)

Firstly, we show the ability of $\mathcal{T}$ to compose functions with the assistance of single vectors.

Theorem 4.11. *Any regular 3×3 matrix F can be represented by $\mathcal{T}$ and single qutrits.*

Proof. F can be represented by LDU decomposition as

$$F = PLDUP' \tag{4.25}$$

P and P' are permutations, L is a lower triangle matrix, U is an upper triangle matrix, and D is a diagonal matrix. Moreover, the diagonal elements of L and U are all 1. In other words, L, U, and D are

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_2 & 1 & 0 \\ l_1 & l_0 & 1 \end{pmatrix} \tag{4.26}$$

$$D = \begin{pmatrix} d_0 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix} \tag{4.27}$$

$$U = \begin{pmatrix} 1 & u_0 & u_1 \\ 0 & 1 & u_2 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.28}$$

We define $|\psi\rangle$, $|\phi\rangle$, $|\pi\rangle$, $|\eta\rangle$, and $|\zeta\rangle$ as

$$|\psi\rangle := d_0|0\rangle + d_1|1\rangle + d_2|2\rangle \tag{4.29}$$

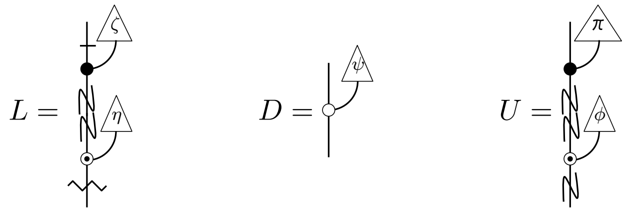
$$|\phi\rangle := (u_2 - u_0)|0\rangle + |1\rangle + |2\rangle \tag{4.30}$$

$$|\pi\rangle := u_1|0\rangle + u_0|1\rangle + |2\rangle \tag{4.31}$$

$$|\eta\rangle := (l_2 - l_0)|0\rangle + |1\rangle + |2\rangle \tag{4.32}$$

$$|\zeta\rangle := l_1|0\rangle + l_0|1\rangle + |2\rangle \tag{4.33}$$

By using these qutrits, L, D, and U are represented as



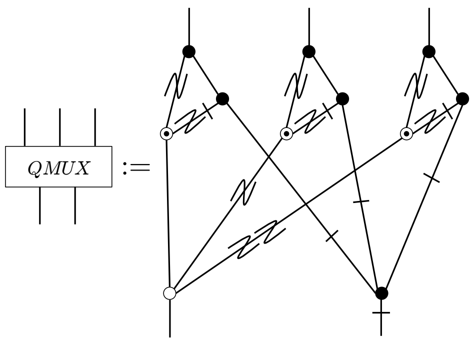
(4.34)

Any permutation can be composited by , , and . □

Therefore, if a qutrit can be composed by $\mathcal{T}$ and single vectors, then all qutrits SLOCC-equivalent to a qutrit can be composed by $\mathcal{T}$ and single vectors.

Moreover, $\mathcal{T}$ can "bring up" qutrits. [17] defined a "quantum multiplexer" of qubits which they called QMUX. We define the qutrit version of QMUX.

Definition 4.12 (QMUX). *A QMUX (quantum multiplexer) is*

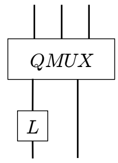


(4.35)

Theorem 4.13. *$\mathcal{T}$ can compose a function which converts single qutrits $|\psi\rangle \otimes |\phi\rangle \otimes |\zeta\rangle$ to $|0\psi\rangle + |1\phi\rangle + |2\zeta\rangle$.*

Proof. Tracing each line, we can check that QMUX converts single qutrits $|\psi\rangle \otimes |\phi\rangle \otimes |\zeta\rangle$ to $\langle\phi|2\rangle\langle\zeta|2\rangle|0\psi\rangle + \langle\zeta|2\rangle\langle\psi|2\rangle|1\phi\rangle + \langle\psi|2\rangle\langle\phi|2\rangle|2\zeta\rangle$. Because of Theorem 4.11, there is a regular matrix L such that

$$L = \begin{pmatrix} \frac{1}{\langle\phi|2\rangle\langle\zeta|2\rangle} & 0 & 0 \\ 0 & \frac{1}{\langle\zeta|2\rangle\langle\psi|2\rangle} & 0 \\ 0 & 0 & \frac{1}{\langle\psi|2\rangle\langle\phi|2\rangle} \end{pmatrix} \tag{4.36}$$

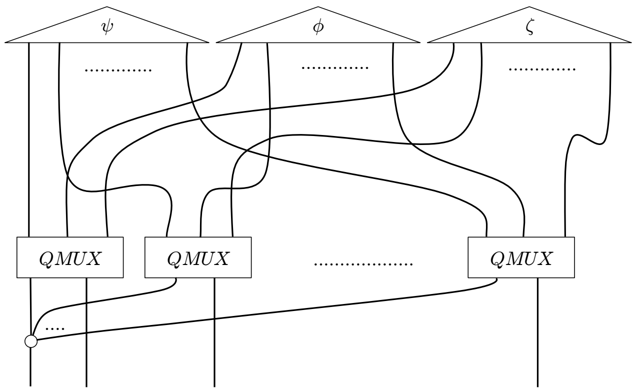


(4.37)

converts $|\psi\rangle \otimes |\phi\rangle \otimes |\zeta\rangle$ to $|0\psi\rangle + |1\phi\rangle + |2\zeta\rangle$. $\square$

Furthermore, aligning QMUXs, this can be extended to the case of N-qutrits.

Theorem 4.14. *Let $|\psi\rangle$, $|\phi\rangle$, and $|\zeta\rangle$ be N-partite qutrits.*



(4.38)

is $|0\psi\rangle + |1\phi\rangle + |2\zeta\rangle$.

Therefore, if any N-qutrit is composed by $\mathcal{T}$ and single vectors, then any N+1-qutrit can be composed by $\mathcal{T}$ and single vectors.

Chapter 5

Related Work

Selinger gave the graphical language on the categorical axioms of quantum protocols in [16]. In the paper, Selinger explicitly showed the graphical representation of quantum entanglement. This was based on the interpretation in [14] of entanglement as name and coname on dagger compact closed categories. Using this interpretation, the flow of quantum information was expressed graphically in [14]. However, this representation was limited to be uniformed, so one could not understand the property of the entanglement.

Graphical representation of how entangled the qubit is was done in [17]. The paper gave graphical representation to tripartite qubits using CFAs. By the representation, they classified SLOCC classes of entangled tripartite qubits. In this paper, we used the way to express tripartite qutrits.

Chapter 6

Conclusion

In this paper, we examined which classes are Frobenius classes in tripartite qutrits. Frobenius states require strong SLOCC-maximality and strong symmetry. In consequence, we showed some classes are not a strongly SLOCC-maximal, and infinite SLOCC classes do not have any symmetric state. We eventually got three Frobenius classes, which classes are SLOCC-equivalent to $|\mathcal{G}\rangle$, $|\mathcal{W}\rangle$, and $|\mathcal{I}\rangle$. After identifying Frobenius classes in tripartite qutrits, we classified them. Two SLOCC classes of them correspond to an SCFA and an ACFA which correspond to tripartite qubits on $\mathbb{C}^2$. For the last of the Frobenius classes, we defined a CFA called an ISCFA. We proved that their correspondences are unique. The classification used the rank of . The equations of an SCFA and an ACFA mean that the ranks of  are $\dim \mathcal{H}$ and 1 respectively. Our definition of an ISCFA requires the rank of  to be neither $\dim \mathcal{H}$ nor 1. The uniqueness implies the algebraic and graphical structure of Frobenius states. In the end, we showed the ability of the three CFAs. They can grow N-partite qutrits and construct any linear function with help of single qutrits. By this construction, we showed any multipartite qutrits can be expressed graphically using graphical representation of CFAs on $\mathbb{C}^3$.

Our way to express qutrits graphically is extension of the way for qubits introduced in [17]. Between result of qutrits and qubits, there are some same points and some different points.

There are three types of the same points. Firstly, in both qutrits and qubits, there are Frobenius states, which are the highly symmetric and highly entangled states. Secondly, an SCFA and an ACFA also exist. Finally, CFAs have the ability to make N-partite same dimensional systems with help of single systems. This ability implies that these CFAs have some kinds of completeness.

On the contrast, there are the different points. Firstly, although four SLOCC classes is not a Frobenius class in tripartite qubits, infinite SLOCC classes are so in tripartite qutrits. Moreover, although all non-symmetric SLOCC classes in tripartite qubits are non-maximal classes, infinite non-symmetric SLOCC classes in tripartite qutrits are not non-maximal classes. Secondly, in qutrits, there is an ISCFA, which is neither an SCFA nor an ACFA. Both points are caused by the higher dimension of qutrits than of qubits. The dimension of qubits is two, which is lowest dimension of entangled states. Because of this lower dimension, qubits do not have any non-symmetric class. The second point is caused by the rank of . In qubits, the rank of a nontrivial function is limited to 1 or $\dim \mathcal{H} = 2$. However, in qutrits, there is an intermediate rank, i.e., 2. Therefore, there is an

ISCFA, which does not exist in qubits.

We showed the correspondence between CFAs and some tripartite qutrits. However, there is an infinite number of SLOCC classes which do not have graphical representation which reflects their entanglement property. The success of Frobenius states implies the algebraic structure of tripartite qutrits. Another algebra is needed to express the rest SLOCC classes graphically specifying their feature. Many classes are not symmetric, so commutative property is not needed for the algebras.

In higher dimensional tripartite systems, Frobenius classes may exist. Considered the rank of , there may be more classification in high dimension. However, the number of these classifications may remain finite, although SLOCC classes are infinite. Also from this point, another graphical representation is needed. Furthermore, in higher dimensional, a question whether an ISCFA does exist or not arises. If an ISCFA exists in another dimension, research into the normal forms of CFAs with ticks and knurls like [20] is also meaningful.

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